Homework set 2

Note: the homework sets are not for submission. They are designed to help you prepare for the quizzes.

1. We are given a binary counter with k bits, that supports the following two operations:

   - Increment: increase the value of the counter by 1 (modulo 2^k).
   - Reset: set the value of the counter to 0.

   Show how to implement the counter as an array of bits, so that any sequence of n Increment and Reset operations takes $O(n)$ time on an initially zero counter. Hint: keep a pointer to the high-order 1.

2. We are given a directed graph $G = (V, E)$, with two special vertices s and t, and arbitrary non-negative capacities $c(e)$ on edges $e \in E$. Additionally, we are given a valid flow f: that is, for each edge $e \in E$, we have a flow value $f(e)$, such that the edge capacity constraints and the flow conservation constraints are satisfied. Moreover, flow f is acyclic: that is, there is no cycle in G on which all edges carry positive flow. Our goal is to find a collection $\mathcal{P}$ of paths connecting s to t, together with values $f'(P) > 0$ for each path $P \in \mathcal{P}$, such that for each edge $e \in E$,

   $$\sum_{\substack{P \in \mathcal{P}: \\ e \in P}} f'(P) = f(e).$$

   Show an efficient algorithm to find such collection of paths with the values $f'(P_i)$ and prove its correctness.

   Remark: Such collection of paths is called a *flow-path decomposition* of the flow f.

3. Let $G = (V, E)$ be an arbitrary directed flow network, with a source s, a sink t, and a positive integer capacity $c(e)$ on every edge $e \in E$. Decide whether the following statement is true or false:

   > Let (A, B) be the minimum s-t cut in G. Suppose we add 1 to the capacity of every edge $e \in E$. Then (A, B) is still a minimum s-t cut with respect to the new capacities.

   If the statement is true, give a proof of its correctness. If it is false, give a counterexample.