Homework set 4

Note: the homework sets are not for submission. They are designed to help you prepare for the quizzes.

1. In this problem we consider the decision version of the shortest path problem: given a directed graph $G = (V, E)$ with integral non-negative edge lengths, two vertices s and t, and an integer k, the goal is to determine whether there is a path of length at most k connecting s to t in G. We denote this problem SP. For each one of the following two statements, decide whether it is true, false, or the answer depends on whether $P = NP$. Give a short explanation.

   (a) $SP \leq_p$ Set Cover.

   (b) $SAT \leq_p SP$.

2. Consider the following solitaire game. The puzzle consists of an $(n \times m)$ grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions: (1) every row contains at least one stone, and (2) no column contains stones of both colors. For some initial configurations of stones, reaching this goal is impossible. Prove that the problem of determining, given an initial configuration of red and blue stones, whether the puzzle can be solved, is NP-complete.

3. Given a graph $G = (V, E)$, a k-coloring is an assignment $f : V(G) \to \{1, \ldots, k\}$ of a color in $\{1, \ldots, k\}$ to each of the vertices of v. A coloring f is *valid*, if for each edge $e = (u, v)$, $f(u) \neq f(v)$. A coloring is *careful*, if for each edge $e = (u, v)$, $|f(u) - f(v)| \geq 2$. Recall that we have shown in class that the problem of determining whether a given graph G has a valid 3-coloring is NP-complete.

   (a) Prove that for each $k > 3$, the problem of determining, given a graph G, whether it has a valid k-coloring, is NP-hard.

   (b) Prove that the problem of determining, given a graph G, whether it has a careful 3-coloring, is in P.

   (c) Prove that the problem of determining, given a graph G, whether it has a careful 4-coloring, is in P.

   (d) Prove that for each $k \geq 5$, the problem of determining, given a graph G, whether it has a careful k-coloring, is NP-hard.

4. We are given a collection of n items, whose weights are $w_1, w_2, \ldots, w_n$. We need to move these items from one city to another in trucks. The maximum amount of weight that each truck is allowed to carry is K. We would like to move all these items using the minimum possible number of trucks.

   Consider the following simple algorithm. Start loading the items one-by-one into the first truck, until we reach an item whose addition would overflow the weight limit on that truck. We then declare the first truck full and send it off. We then continue to the second truck with the remaining items, and so on.

(a) Prove that this algorithm does not always produce an optimal solution to the problem.

(b) Prove that the algorithm always produces a factor-2 approximation.

5. Denote by GREEDY the greedy approximation algorithm we defined in class for the Set Cover problem. Recall that the algorithm chooses at each step a set that covers maximum number of elements that are not covered yet.

   Given a set cover instance I, let s_I denote the maximum set size, $s_I = \max_{S \in I} \{|S|\}$. Show that GREEDY achieves an $O(\log(s_I))$-approximation.

   Hint: Partition the algorithm into phases, according to $\max_{S \in I} \{|S \cap U'|\}$ where U' is the set of elements that are not covered yet.