Homework Assignment 3

Due: Tuesday, June 3 in class.

There will be no extensions!

General remark: Whenever you are asked to provide an α-approximation algorithm, you need to prove that your algorithm outputs a feasible α-approximate solution.

You can use any results that were proved in class without having to re-prove them. You can also assume, if needed, that there is an efficient algorithm for computing maximum matching in any graph.

1. **Multicut** In the Minimum Multicut problem, we are given an undirected graph $G = (V, E)$ with weights w_e on edges $e \in E$. Additionally, we are given a collection $\{(s_1, t_1), \ldots, (s_k, t_k)\}$ of pairs of vertices that we would like to disconnect from each other. The goal is to find a minimum-weight subset $E' \subseteq E$ of edges, such that, once we remove the edges of E' from G, no path connects s_i to t_i in the resulting graph, for any $1 \leq i \leq k$. In this question we consider the Minimum Multicut problem on star graphs (trees of height 1).

   (a) Prove that the Minimum Multicut problem is at least as hard to approximate as Vertex Cover, even when the input graph is a star.

   (b) Write a linear programming relaxation for the Minimum Multicut problem on a star graph, and obtain a factor 2-approximation algorithm for the problem via LP-rounding.

   (c) Show that the integrality gap of your LP relaxation for star graphs is at least 2 asymptotically.

2. **Congestion Minimization** Show that the integrality gap of the LP-relaxation of directed Congestion Minimization is at least $\Omega(\log n / \log \log n)$, where n is the number of graph vertices.

   Hint: Use the construction we used in class to lower bound the integrality gap for Machine Minimization.

3. **EDP** Show exact polynomial time algorithm for the special case of the Maximum Edge Disjoint Paths problem, where the input graph is a tree. (Notice that we assume that all edge capacities are 1). Prove the algorithm's correctness.

   Hint: Use dynamic programming.

4. **Set Cover** Recall that in class we have seen a primal-dual algorithm for the Set Cover problem, whose approximation factor is f-the maximum frequency of any element (that is, the number of sets containing the element).

   (a) Prove that the integrality gap of the LP-relaxation we have used for the Set Cover problem is at most f.

   (b) Prove that the integrality gap of this relaxation is $\Omega(f)$, for any choice of the frequency value f. Hint: define an element for every f-tuple of sets.

   (c) Does this lower bound rule out obtaining a better than factor $\Theta(f)$-approximation to Set Cover using the primal-dual method? Explain your answer.

5. **Multiway Cut** Consider the following generalization of the Multiway Cut problem. We are given an undirected graph $G = (V, E)$ with costs $w_e \geq 0$ on edges, and a subset $T = \{t_1, \ldots, t_k\}$ of vertices called terminals. Additionally, each vertex $v \in V$ has a list $L(v) \subsetneq T$ of *forbidden* terminals. The goal is to partition the graph vertices into k subsets $C_1, \ldots, C_k$, such that for all $i : 1 \leq i \leq k$, $t_i \in C_i$, and each vertex $v \in V$ belongs to some cluster C_j with $j \notin L(v)$. We need to minimize the total cost of edges e whose endpoints belong to distinct subsets C_i.

   (a) Extend the LP relaxation for the multiway cut problem to this generalized setting.

   (b) Consider the following LP-rounding algorithm. At the beginning of the algorithm, all the vertices are unassigned. While there is an unassigned vertex, perform the following procedure:

   > Procedure Partition
   >
   > For $i = 1, 2, \ldots, k$:
   >
   > - Choose $r_i \in (0, 1)$ uniformly at random.
   > - For each unassigned vertex v with $x_i(v) \geq r_i$, assign v to C_i.

      i. Prove that the above algorithm produces a feasible solution for the generalized multiway cut problem.

      ii. Prove that the algorithm achieves an $O(\log n)$-approximation.
      Hint: Bound the number of calls to Procedure Partition and bound the expected cost of each execution of the procedure.

   (c) Design a factor 4/3-approximation algorithm for generalized multiway cut with $k = 3$ terminals. (Partial credit will be given for constant factor approximation algorithm).