Homework Assignment 1

Due: Wednesday, April 27 in class.

Homework Policy: You are free to discuss the problems with other students and consult online material. However, you must write up your own solutions in your own words and mention the names of the people you discussed them with, and sources you consulted.

You can use any results that were proved in class.

1. Prove that the family of all graphs of treewidth at most k is minor-closed.
2. Prove that a graph G has treewidth 1 if and only if G is a forest.
3. Prove or disprove: for every pair G, H of graphs, such that the maximum vertex degree in H is at most 4, H is a minor of G if and only if H is a topological minor of G.
4. Prove that graph G on n vertices has treewidth $n-1$ if and only if G is a complete graph.
5. Given a graph $G=(V,E)$, a subset $S \subseteq V$ of vertices is a *vertex cover* for G, iff for every edge $e \in E$, at least one endpoint of e belongs to S. The goal in the Minimum Vertex Cover problem is to find a vertex cover $S \subseteq V$ of minimum cardinality. Let $VC(G)$ denote the value of the optimal solution to this problem.
   (a) Show that the treewidth of G is no more than $VC(G)$.
   (b) Design a fixed-parameter tractable algorithm for Vertex Cover. The running time of the algorithm should be $O(f(k) \cdot \text{poly}(n))$, where $k = VC(G)$, $n = |V|$, and f is some function.
6. In the Minimum Dominating Set problem, we are given a graph $G=(V,E)$ with weights $w_v \geq 0$ on vertices $v \in V$. A subset $S \subseteq V$ of vertices is called a *dominating set* iff for every vertex $v \in V$, either $v \in S$, or at least one of the neighbors of v belongs to S. The goal is to find a dominating set S with minimum weight $\sum_{v \in S} w_v$. Design a PTAS for the Minimum Dominating Set problem in planar graphs. Prove the algorithm's correctness and analyze its running time. Hint: use Baker's technique.