Homework Assignment 3

Due: Wednesday, June 1 in class.
There will be no extensions!

Homework Policy: You are free to discuss the problems with other students and consult online material. However, you must write up your own solutions in your own words and mention the names of the people you discussed them with, and sources you consulted.

You can use any results that were proved in class.

1. Suppose we are given a path P with integral capacities $c_e \geq 0$ for all edges $e \in E(P)$. We are also given a set $\mathcal{R} = \{R_1, R_2, \ldots, R_k\}$ of demand requests. Each request R_i consists of a pair of vertices (u_i, v_i), and asks for 1 flow unit on the path $P_i \subseteq P$ connecting u_i to v_i. Additionally, each request R_i has a profit $w_i > 0$. The goal is to compute a maximum-profit subset $\mathcal{R}' \subseteq \mathcal{R}$ of feasible requests; a set $\mathcal{R}'$ of requests is feasible if, for every edge e, the total number of requests $R_i \in \mathcal{R}'$ whose path contains e is at most c_e.

   (a) Write a (natural) LP-relaxation for the problem, with a variable x_i for each request R_i.

   (b) Show that for any basic feasible solution x to the LP-relaxation, there is some value i, such that $x_i \in \{0, 1\}$. Hint: Let x be a basic feasible solution, where all values $x_i \in (0, 1)$. Consider a set E' of edges corresponding to tight constraints, whose constraints are linearly independent, and $|E'| = k$. Assume first that $E(P) = E'$. Reach a contradiction by showing that for every edge $e = (u, v) \in E'$, at least two requests must start or end at u, and at least two requests must start or end at v. In order to do so, exploit the linear independence of the constraints, the fact that edge capacities are integral, while values x_i are not. Extend this argument to the general case where $E(P) \neq E'$.

   (c) Show an efficient LP-rounding algorithm to find an optimal solution to the problem.

2. Let $G = (V_1, V_2, E)$ be a bipartite graph, and assume that it is k-regular (that is, every vertex is incident on exactly k edges). Show that there is an efficient algorithm to partition $E(G)$ into subsets $E_1, E_2, \ldots, E_k$, such that for all $1 \leq i \leq k$, E_i is a matching. Hint: use the LP-rounding algorithm for maximum-weight bipartite matching we saw in class.

3. Suppose we are given a hypergraph G, and a coloring of its vertices by two colors. We say that an edge e of G is monochromatic iff all vertices of e have the same color. Show that if every edge of G contains exactly $k \geq 4$ vertices, and every vertex participates in at most $r \leq \frac{2^{k-1}}{8k}$ edges, then there is a coloring of the vertices of G by two colors, in which no edge is monochromatic.

4. Let G be a directed graph, where every vertex has in-degree and out-degree exactly $k \geq 8$. The goal of this problem is to show that G contains at least N disjoint cycles, where $N = \left\lfloor \frac{k}{3 \ln k} \right\rfloor$, using Lovasz Local Lemma.

   (a) Suppose we partition the vertices of $G = (V, E)$ into N subsets $S_1, \ldots, S_N$, where every vertex chooses independently uniformly at random one of the subsets to be added to. For every vertex v, let A_v be the event that there is no vertex u in the same set as v with $(v, u) \in E(G)$. Show that $\mathbf{Pr}\left[A_v\right] \leq 1/k^3$.

(b) Let $Z = \{z_u \mid u \in V\}$ be a set of random variables, where the value of variable z_u is i iff vertex u is added to set S_i. What is the set $\mathrm{vbl}(A_v)$ for $v \in V$?

(c) Prove that for all $v \in V$, $|\Gamma(v)| \leq (k+1)^2$.

(d) Prove that G contains at least N node-disjoint cycles.