Graph Routing Problems: Approximation, Hardness, and Graph-Theoretic Insights

Julia Chuzhoy

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ROAD
CLOSED
DETOUR


VLSI
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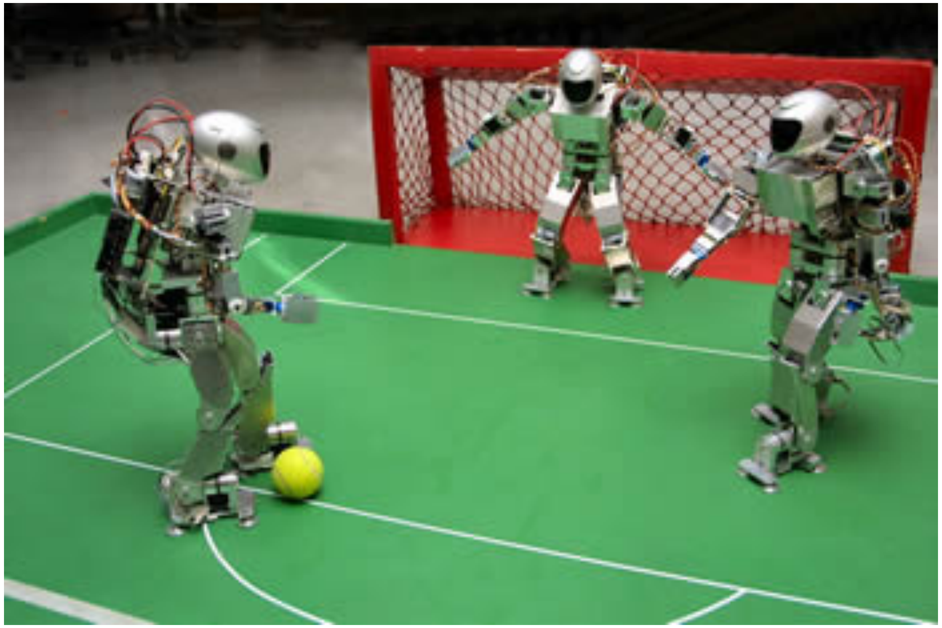


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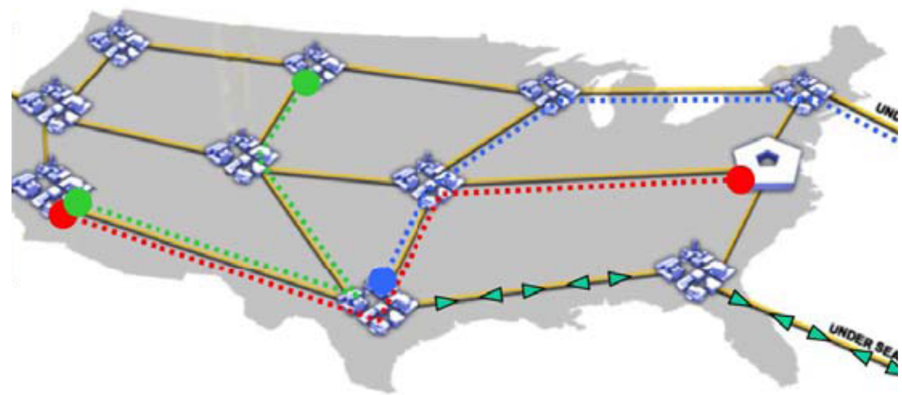


Graph Routing Problems

maximum s-t flow

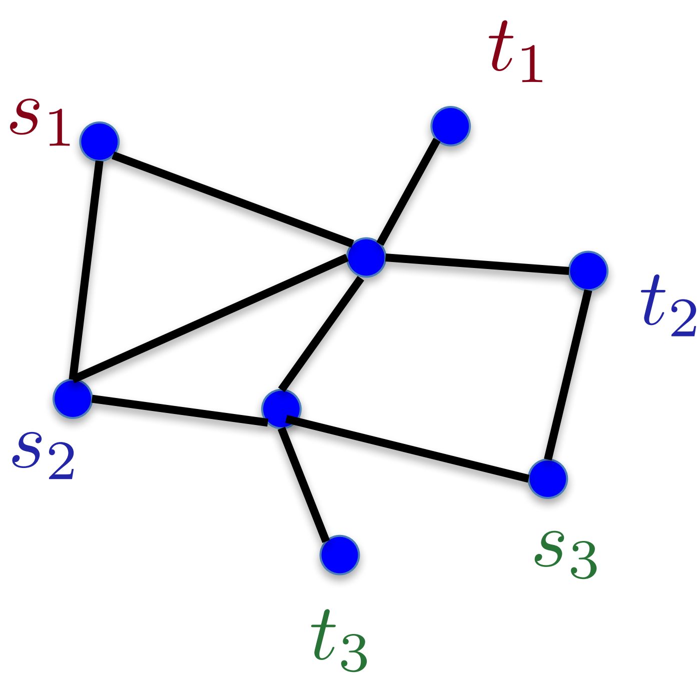
maximum multicommodity flow

maximum node-disjoint paths (NDP)

Node-Disjoint Paths (NDP)

Input: Graph G, source-sink pairs $(s_1,t_1),\ldots,(s_k,t_k)$.

Goal: Route as many pairs as possible via node-disjoint paths



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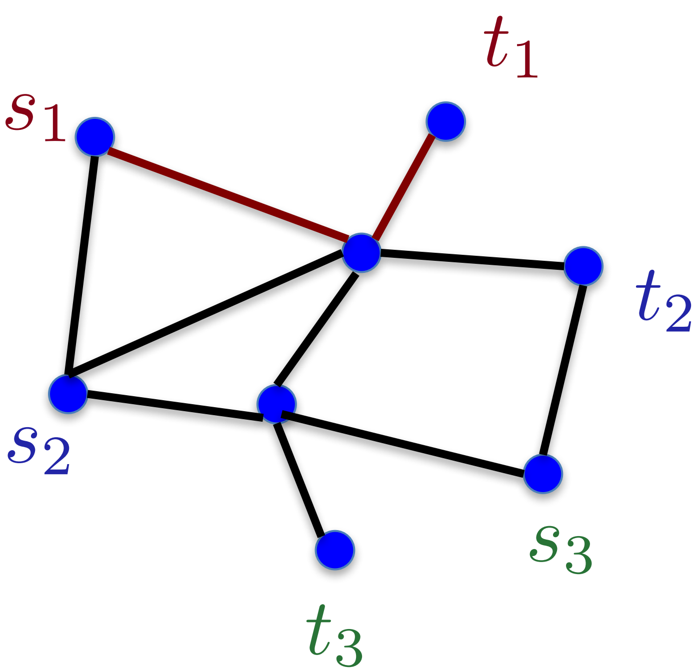

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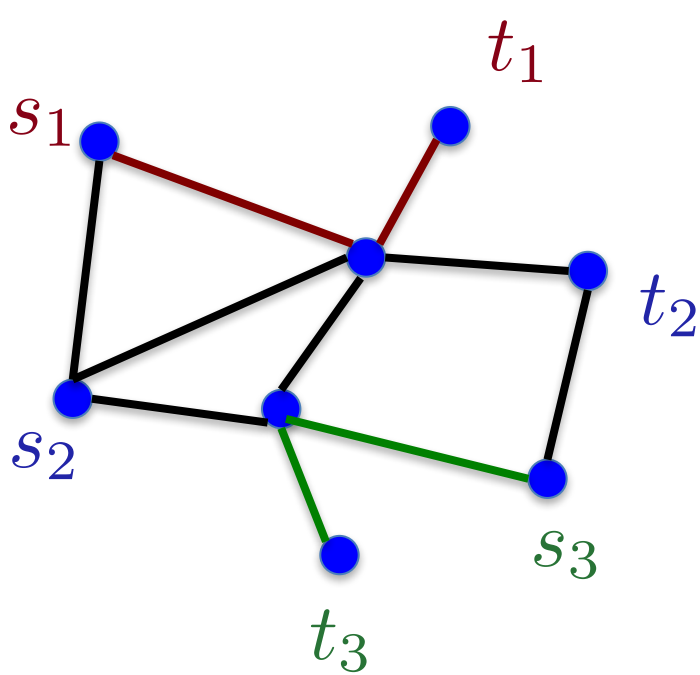

Solution value: 2

OPT: value of best possible solution

Node-Disjoint Paths (NDP)

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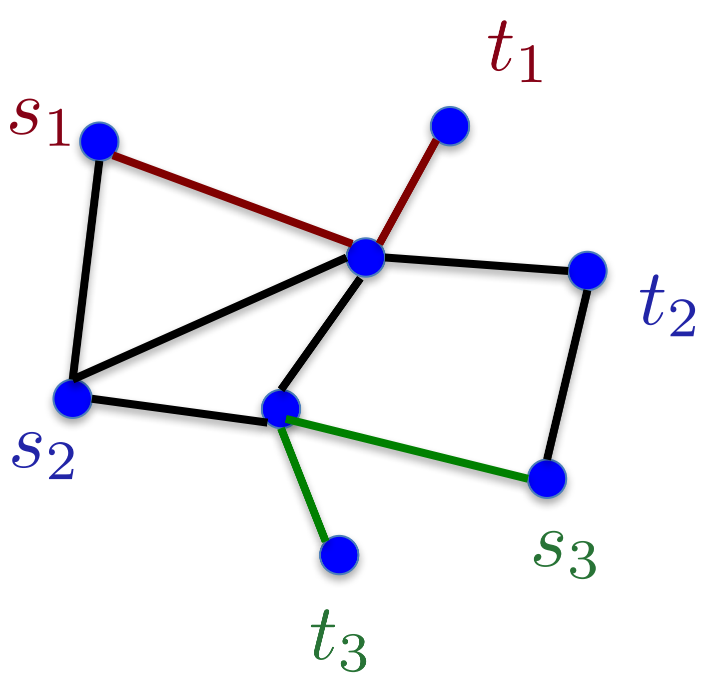
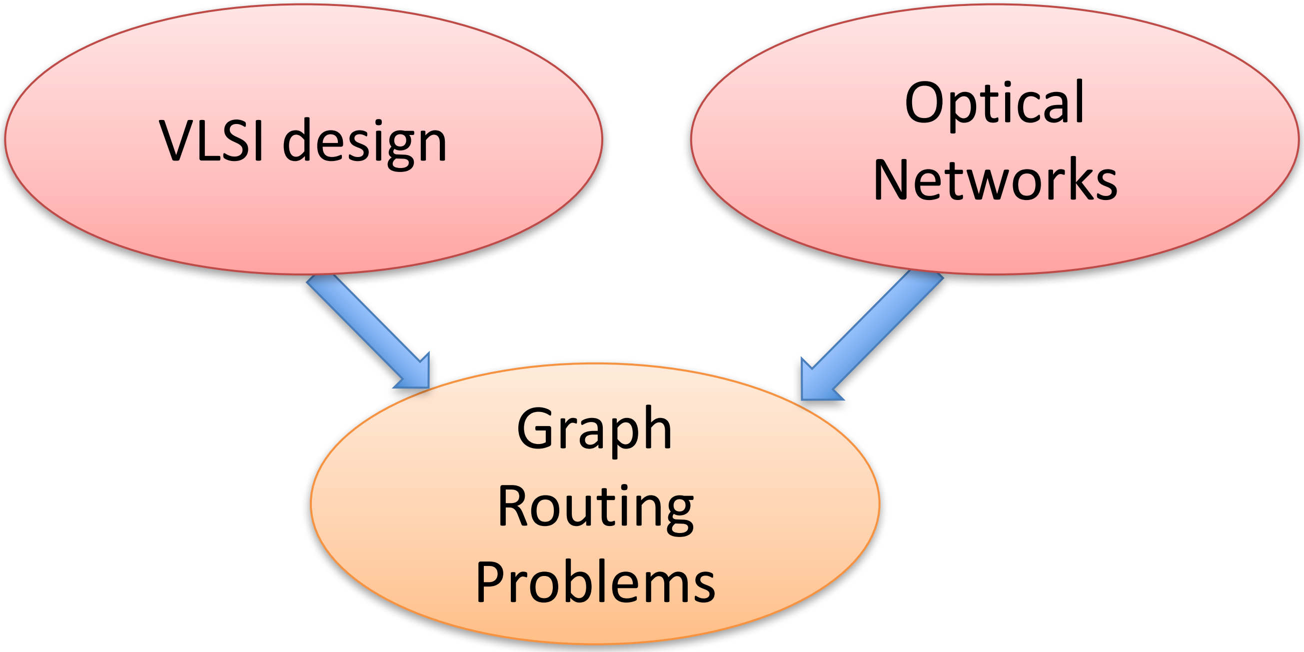
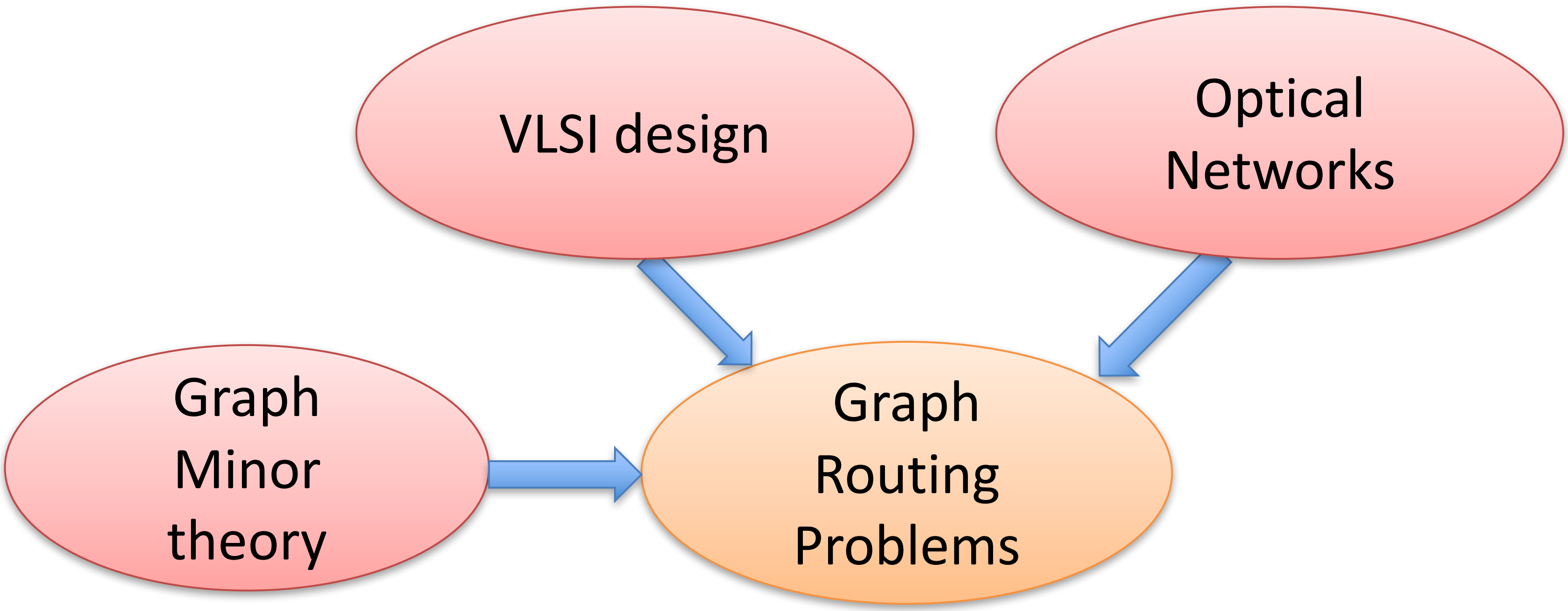


Solution value: 2

Edge-disjoint Paths (EDP): paths must be edge-disjoint


VLSI design
Optical Networks
Graph Routing Problems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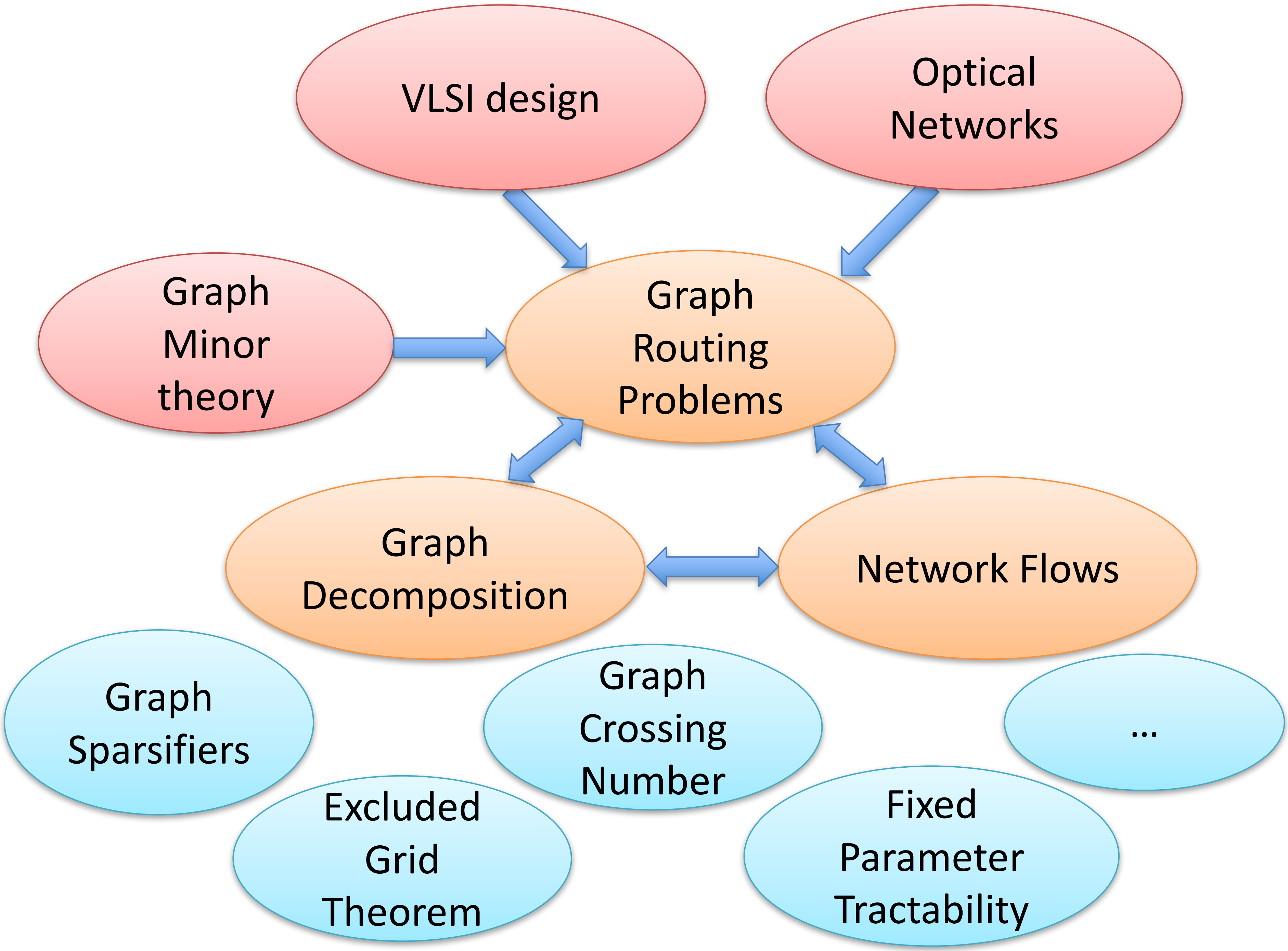
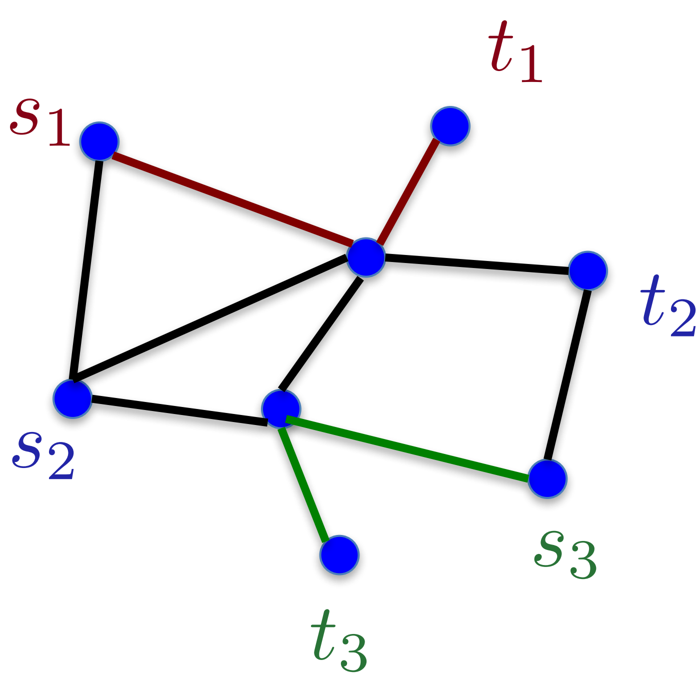

VLSI design
Optical Networks
Graph Minor theory
Graph Routing Problems
Graph Decomposition
Network Flows
Graph Sparsifiers
Graph Crossing Number
...
Excluded Grid Theorem
Fixed Parameter Tractability

Node-Disjoint Paths (NDP)

Input: Graph G, source-sink pairs $(s_1,t_1),\ldots,(s_k,t_k)$.

Goal: Route as many pairs as possible via node-disjoint paths

terminals

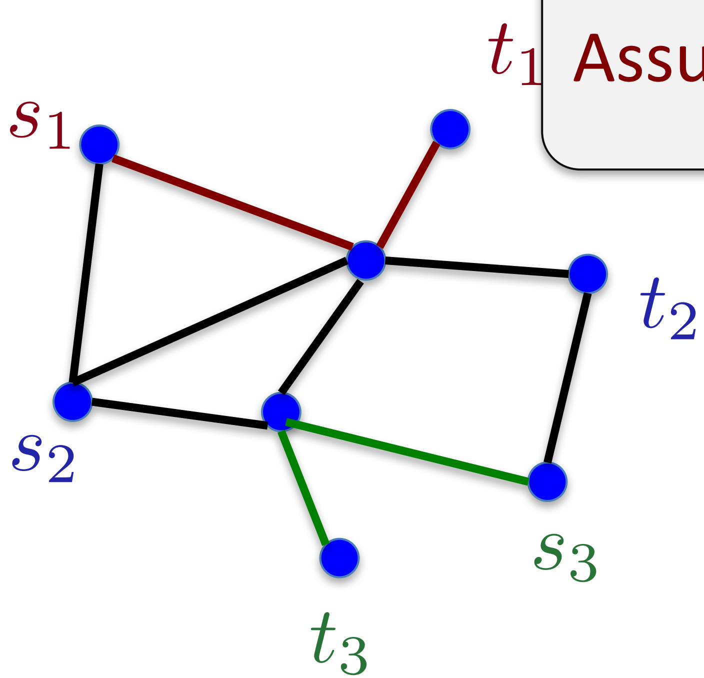


Node-Disjoint Paths (NDP)

Input: Graph G, source-sink pairs $(s_1,t_1),\ldots,(s_k,t_k)$.

Goal: Route as many pairs as possible via node-disjoint paths

Assumption: All terminals are distinct



n – number of graph vertices

k – number of demand pairs

Node-Disjoint Paths (NDP)

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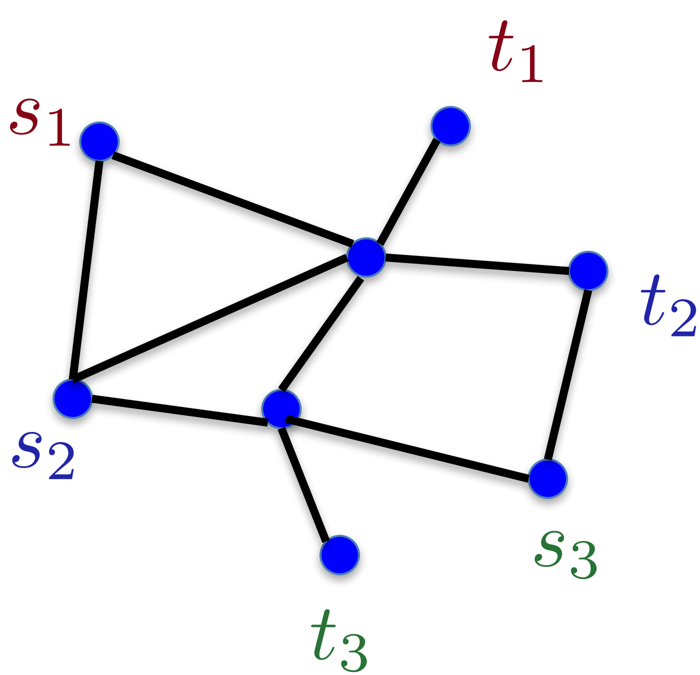


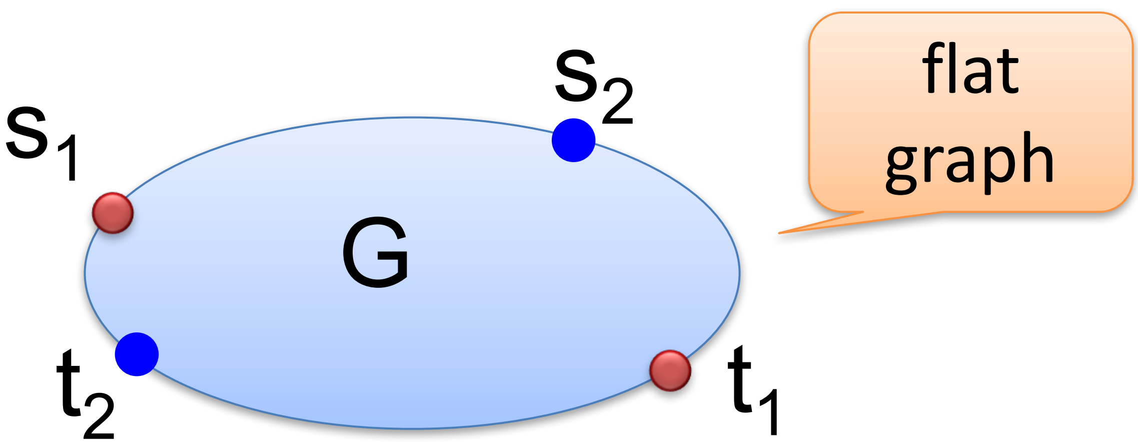
Can we solve it efficiently?

k=1? ✔

k=2?

NDP with k=2

- NP-hard in directed graphs [Fortune, Hopcroft, Wyllie '80]
- Efficiently solvable in undirected graphs [Jung ‘70, Shiloach ’80, Thomassen ‘80, Robertson-Seymour ’90]



Larger k?

Larger k?

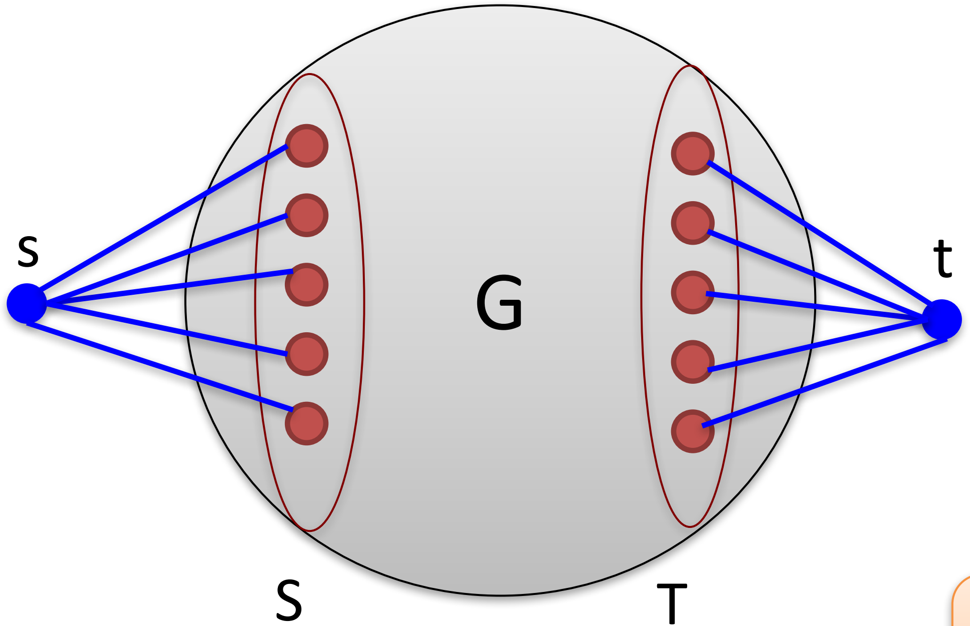
- Constant k: efficiently solvable [Robertson, Seymour '90]
  - Running time: f(k)•n^2 [Kawarabayashi, Kobayashi, Reed '12]

$$f(k) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{k}}}}}}$$

Larger k?

- Constant k: efficiently solvable [Robertson, Seymour ’90]
  - Running time: $f(k) \cdot n^2$ [Kawarabayashi, Kobayashi, Reed ‘12]
- NP-hard when k is part of input [Knuth, Karp ’74]

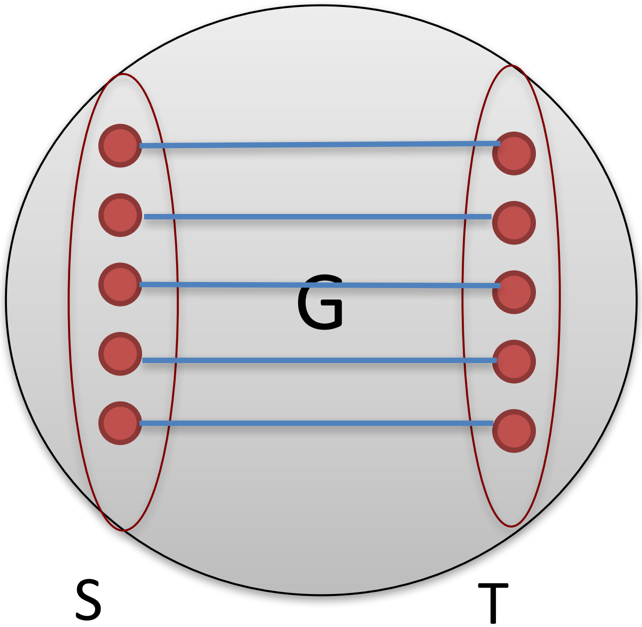
Example



Max s-t flow

- Set of demand pairs is SxT: can solve efficiently

Example



- Set of demand pairs is SxT: can solve efficiently
- Demand pairs are a specific matching between S and T: NP-hard

Dealing with NP-Hardness

An α-approximation algorithm:

- efficient algorithm
- always produces solutions routing at least OPT/α demand pairs.

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optimum solution value

On Approximation Factors

- A simple way to compare algorithms
  - $\alpha=1+\varepsilon$
  - $\alpha=2$
  - $\alpha=O(\log n)$
  - $\alpha=O(\sqrt{n})$
  - …

On Approximation Factors

- A simple way to compare algorithms
- Design algorithms with good approximation factors α
- Establish best possible approximation factor α for a given problem

Hardness of approximation results

On Approximation Factors

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Goal: powerful, simple algorithmic techniques with provable bounds

On Approximation Factors

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Better models for real-life problems

Understanding what makes a problem difficult

On Approximation Factors

- A simple way to compare algorithms
- Design algorithms with good approximation factors α
- Establish best possible approximation factor α for a given problem

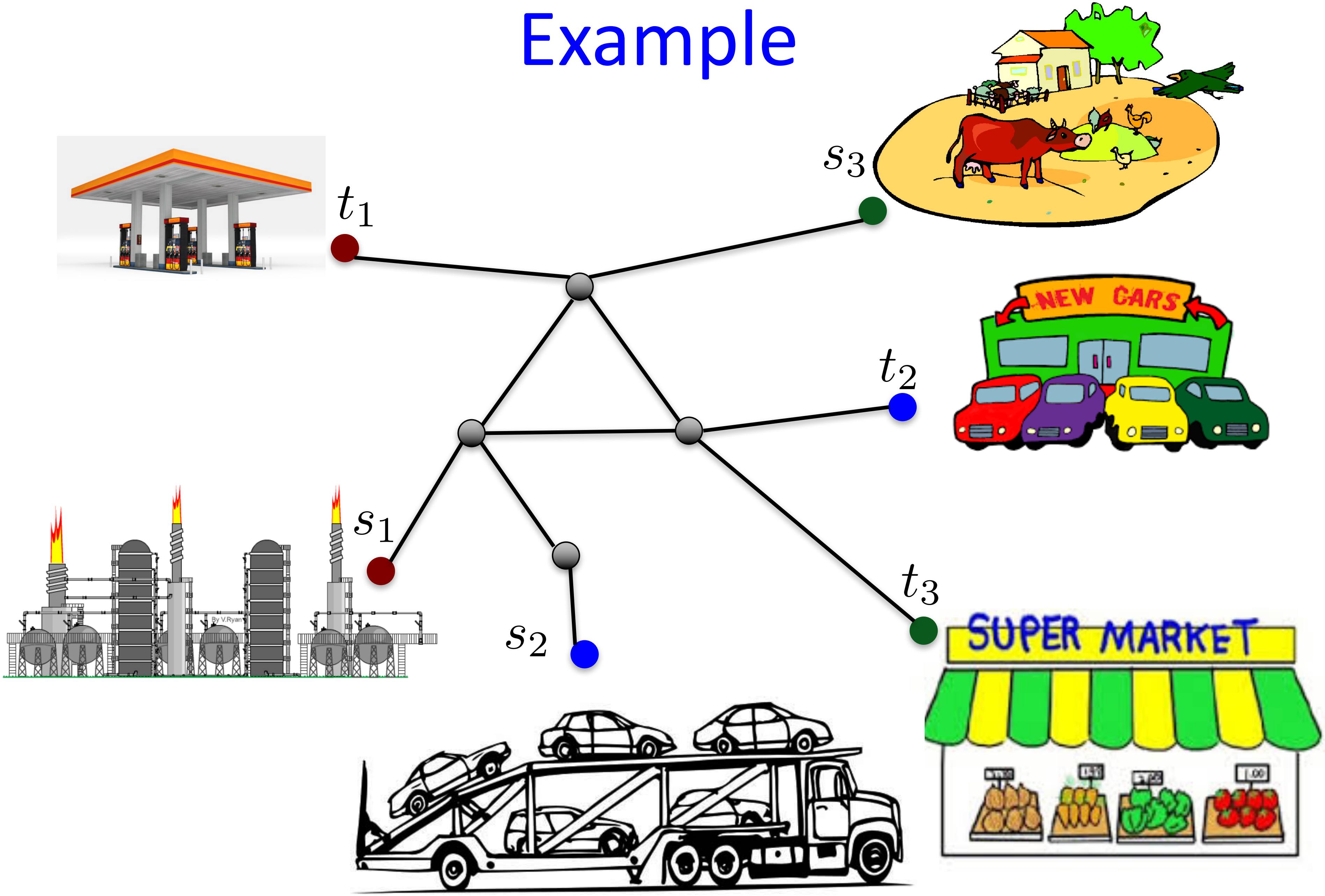
Dealing with NP-Hardness

An α-approximation algorithm:

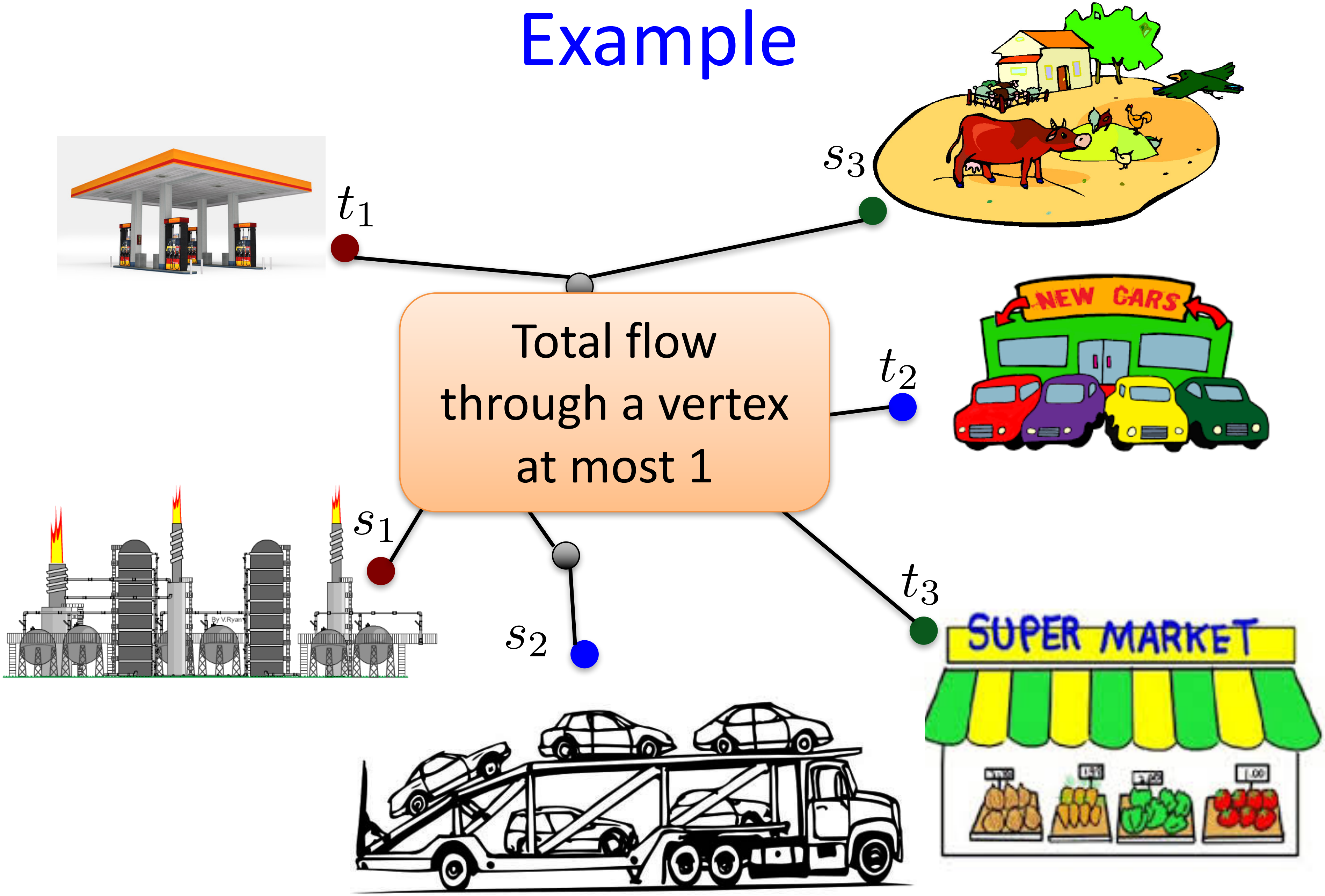
- efficient algorithm
- always produces solutions routing at least OPT/α demand pairs.

Multicommodity Flow relaxation: send as much flow as possible between the s_i-t_i pairs.

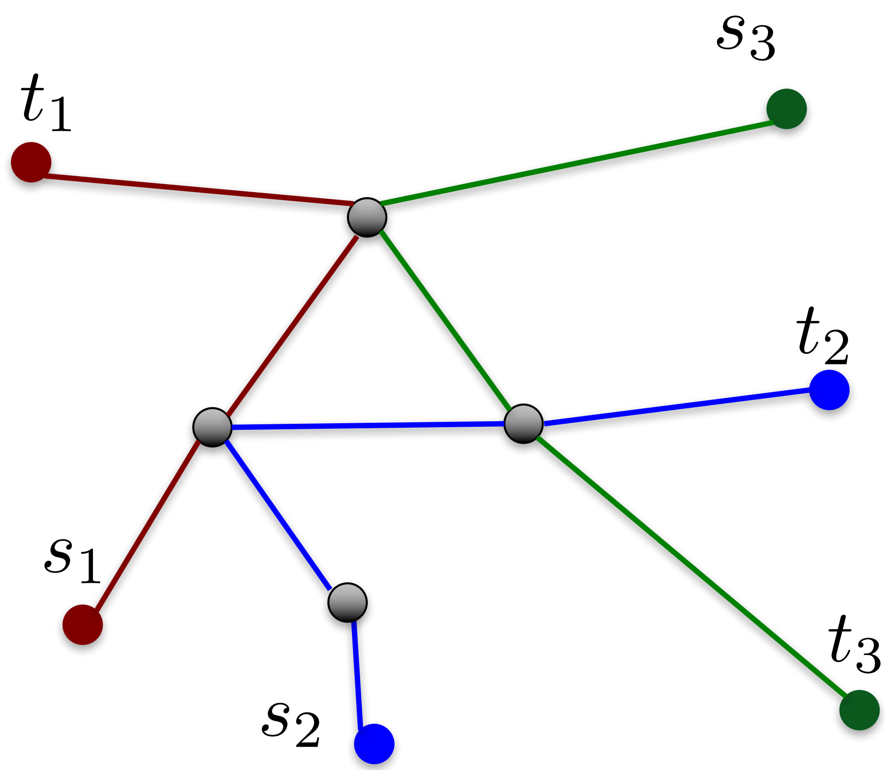
Example



Example


s_3
t_1
NEW CARS
Total flow through a vertex at most 1
t_2
s_1
s_2
t_3
SUPER MARKET

Example



NDP solution value: 1

- send ½ flow unit on each of the 3 paths
- solution value: 3/2

Multicommodity Flows

- Can be computed efficiently
- $OPT_{flow} \geq OPT$

multicommodity flow LP-relaxation

fractional solution

integral solution

Multicommodity Flows

- Can be computed efficiently

multicommodity flow LP-relaxation

- $OPT_{flow} \geq OPT$

- Use the flow to find integral routing of at least OPT_{flow}/α demand pairs

α-approximation algorithm

LP-rounding technique

Approximation Algorithm [Kolliopoulos, Stein '98]

While there is a path P with f(P)>0:

- Add such shortest path P to the solution
- For each path P' sharing vertices with P, set f(P') to 0

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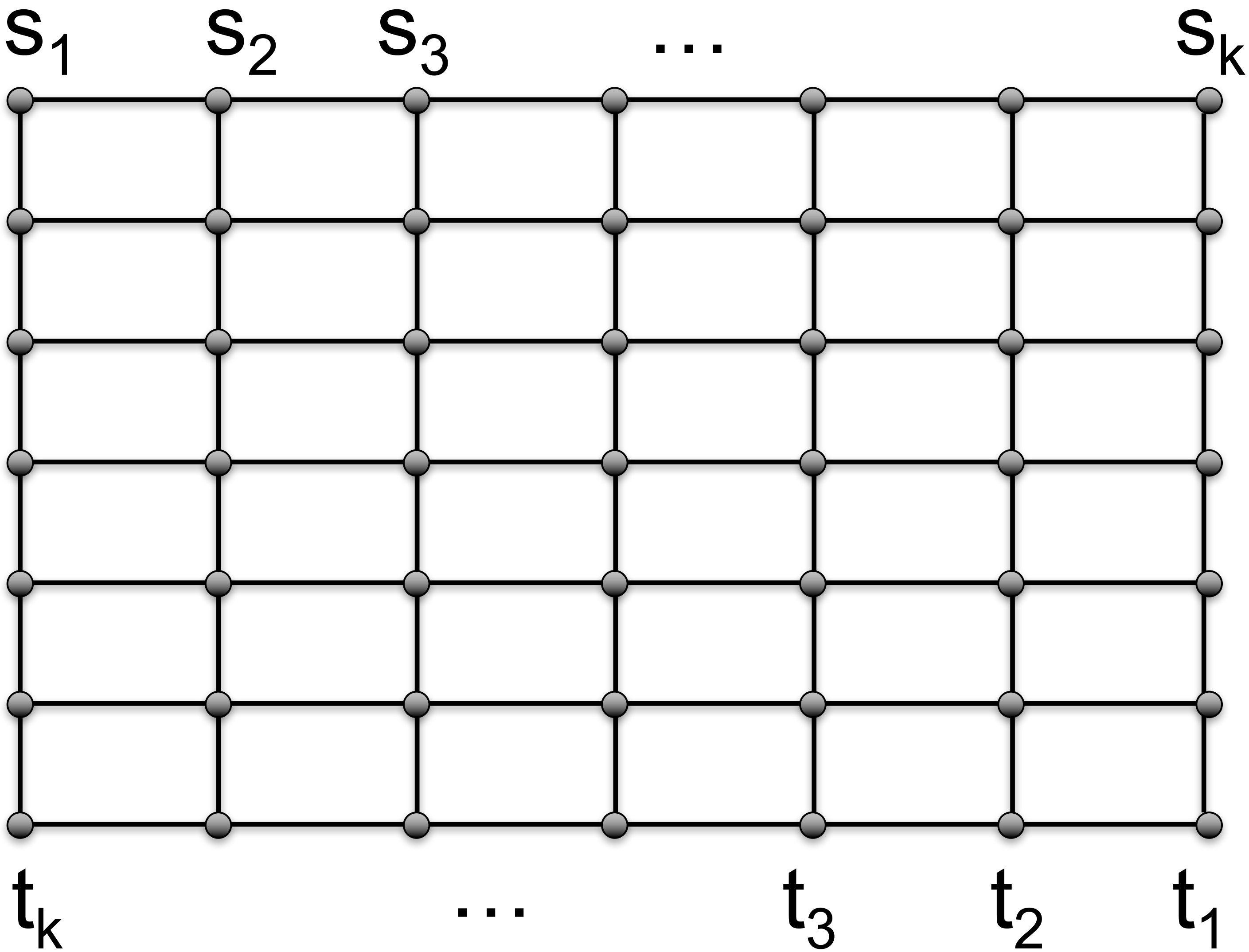
$O(\sqrt{n})$-approximation

Can We Do Better?

- Not if we use the maximum multicommodity flow approach!

Bad Example



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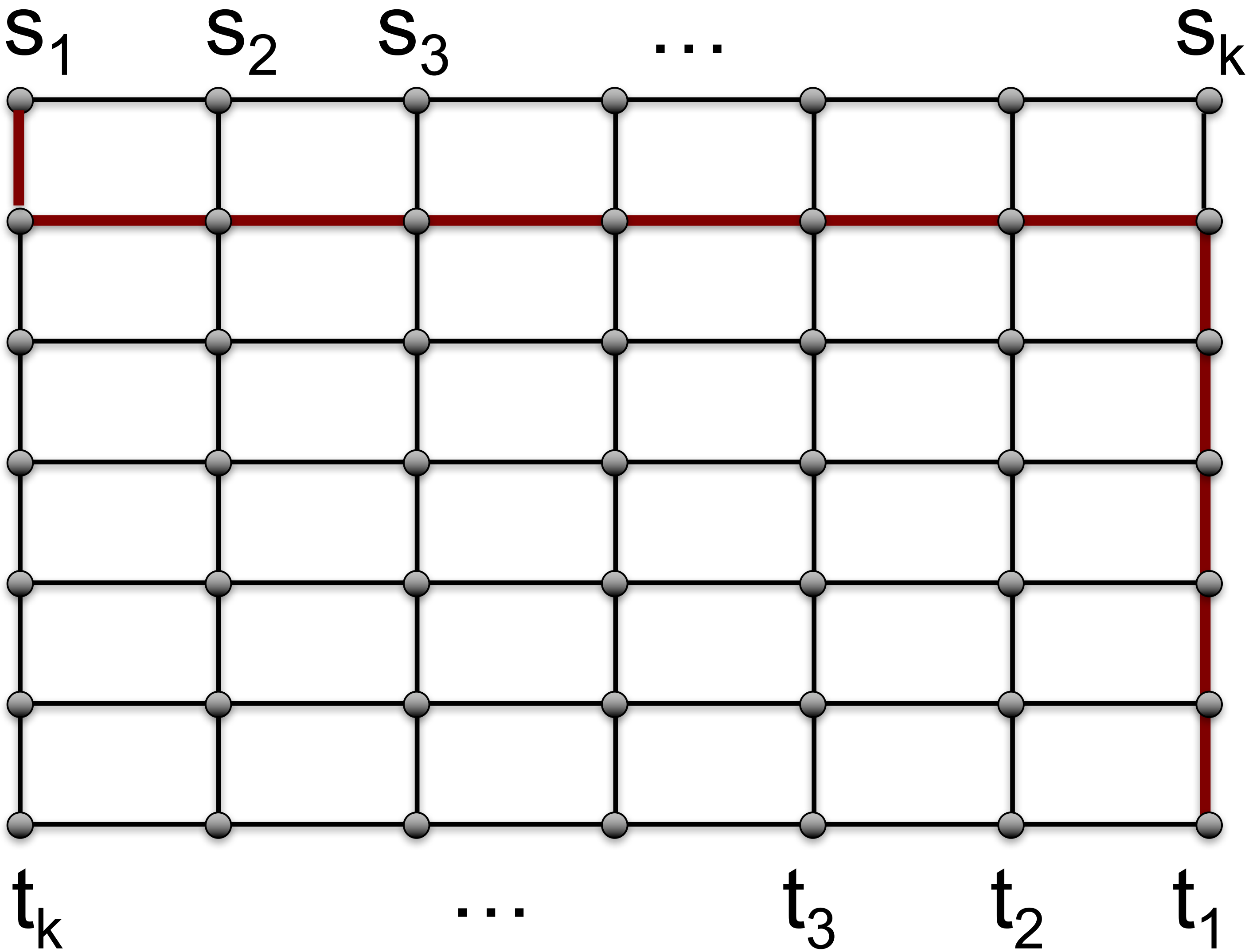

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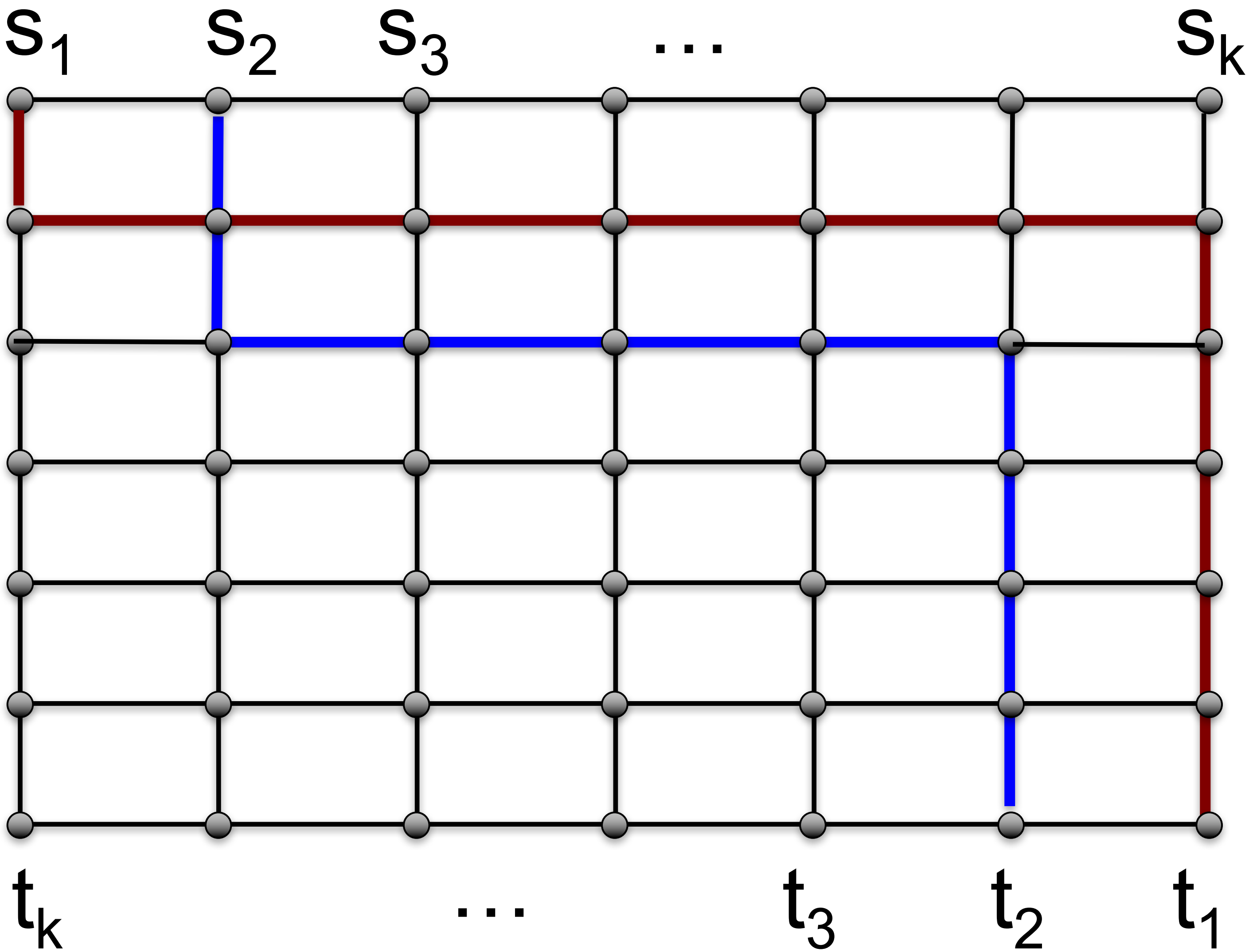

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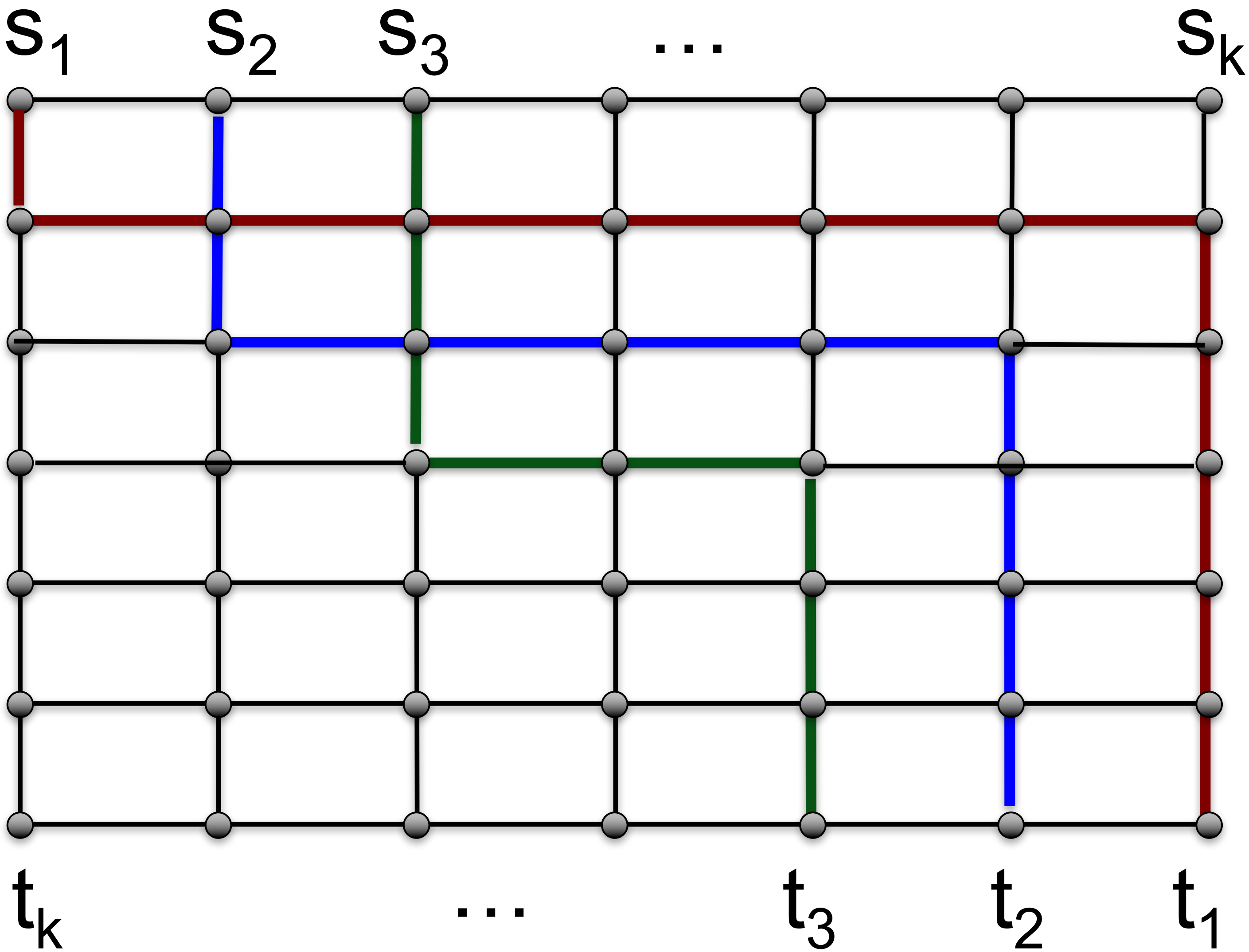

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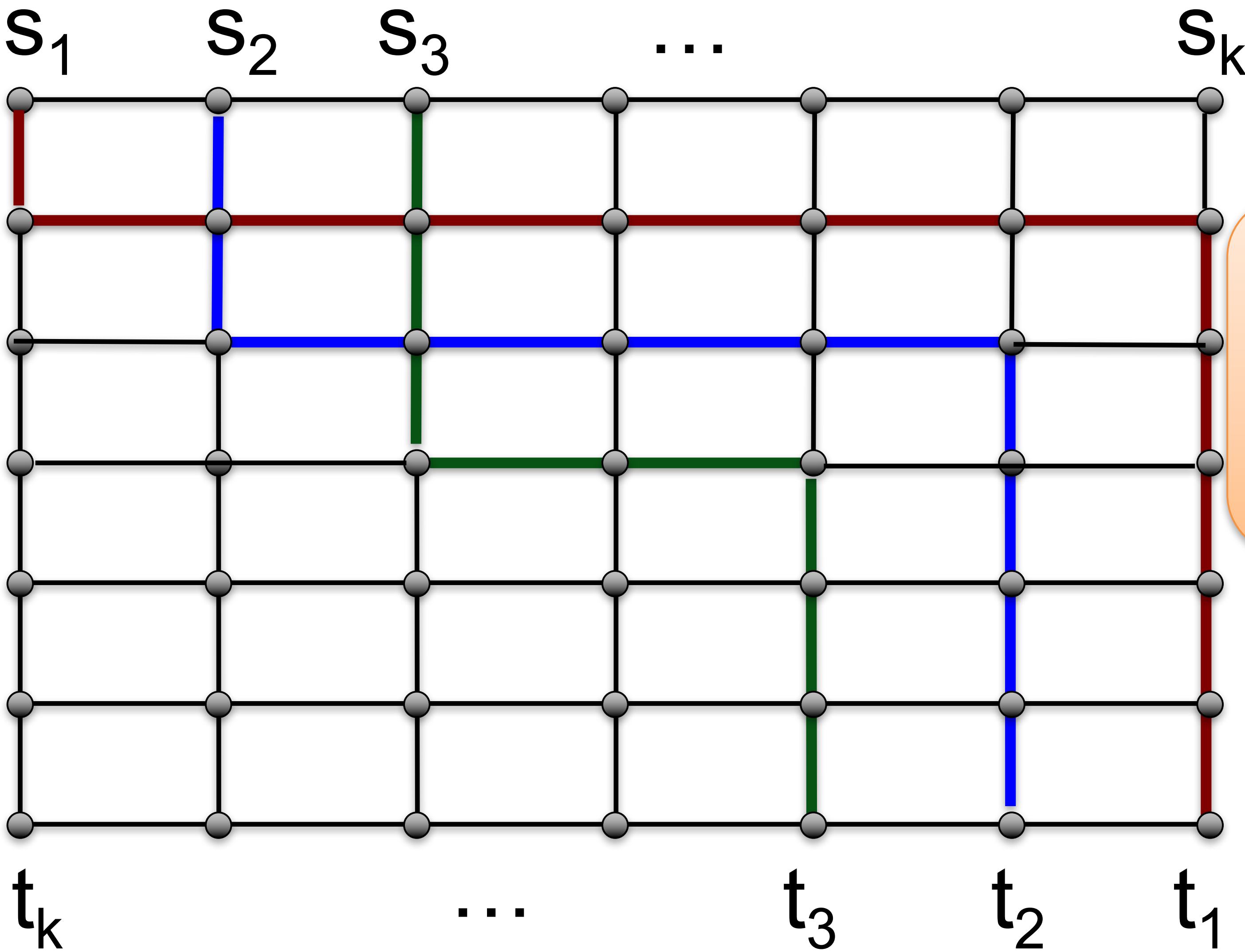


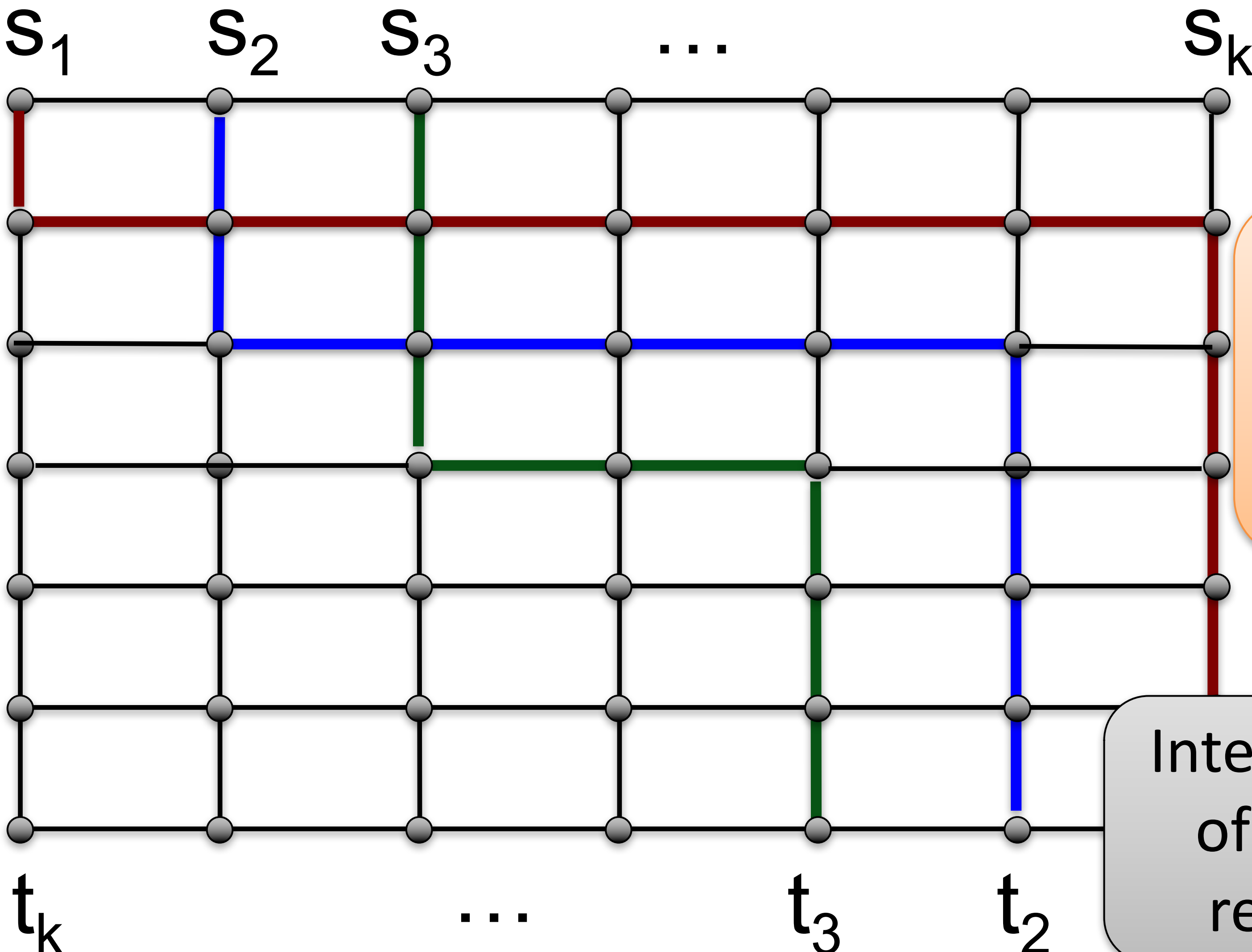
OPT_{flow}=k/3

OPT=1

gap:

$\Omega(k) = \Omega(\sqrt{n})$

Bad Example



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Integrality gap of the flow relaxation

Can We Do Better?

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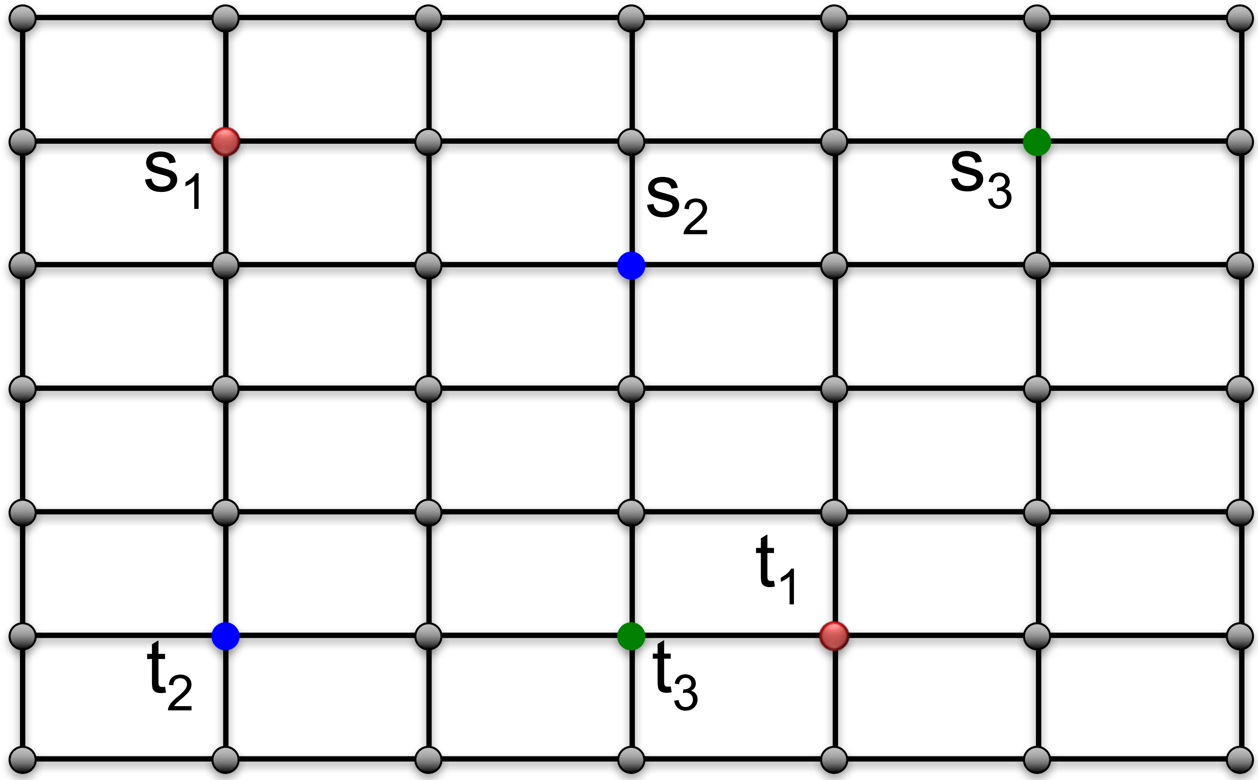
- $\Omega(\log^{1/2-\epsilon} n)$-hardness of approximation for any ϵ [Andrews, Zhang '05], [Andrews, C, Guruswami, Khanna, Talwar, Zhang '10]

Approximation Status of NDP

- $O(\sqrt{n})$ -approximation algorithm
  - even on planar graphs
  - even on grid graphs
- $\Omega(\log^{1/2-\epsilon} n)$ -hardness of approximation for any ϵ [Andrews, Zhang '05], [Andrews, C, Guruswami, Khanna, Talwar, Zhang '10]

Only NP-hardness known for planar graphs and grids

NDP in Grids



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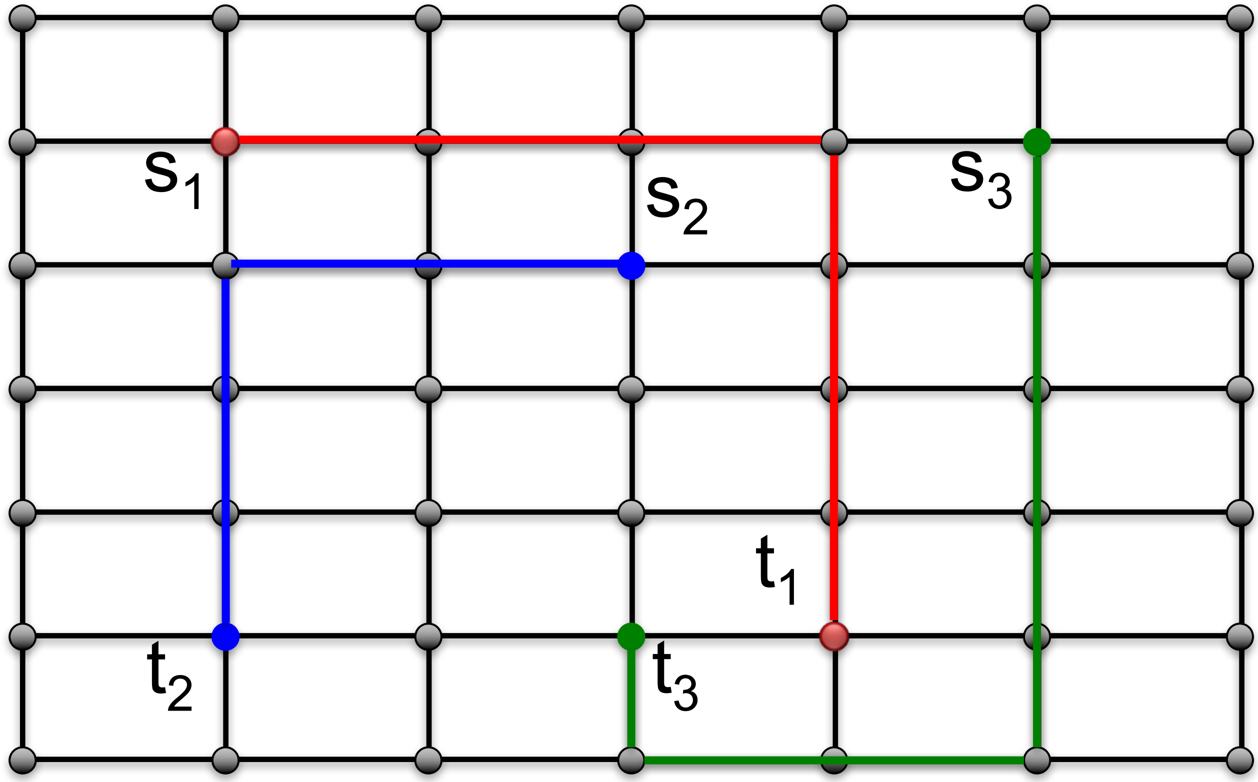

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Until recently:

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[C, Kim '15]
$\tilde{O}(n^{1/4})$-approximation

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$2^{\Omega(\sqrt{\log n})}$-hardness of approximation for subgraphs of grids

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Until recently:

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- $\Omega(\log$ hardness o

[Andrews, Zhang '05], [Andrews, C, Guruswami, Khanna, Talwar, Zhang '10]

[C, Kim, Li '16]

$\tilde{O}(n^{9/19})$-approximation

Work in Progress:

Almost polynomial hardness for NDP in grid graphs [C, Kim, Nimavat '17]

on

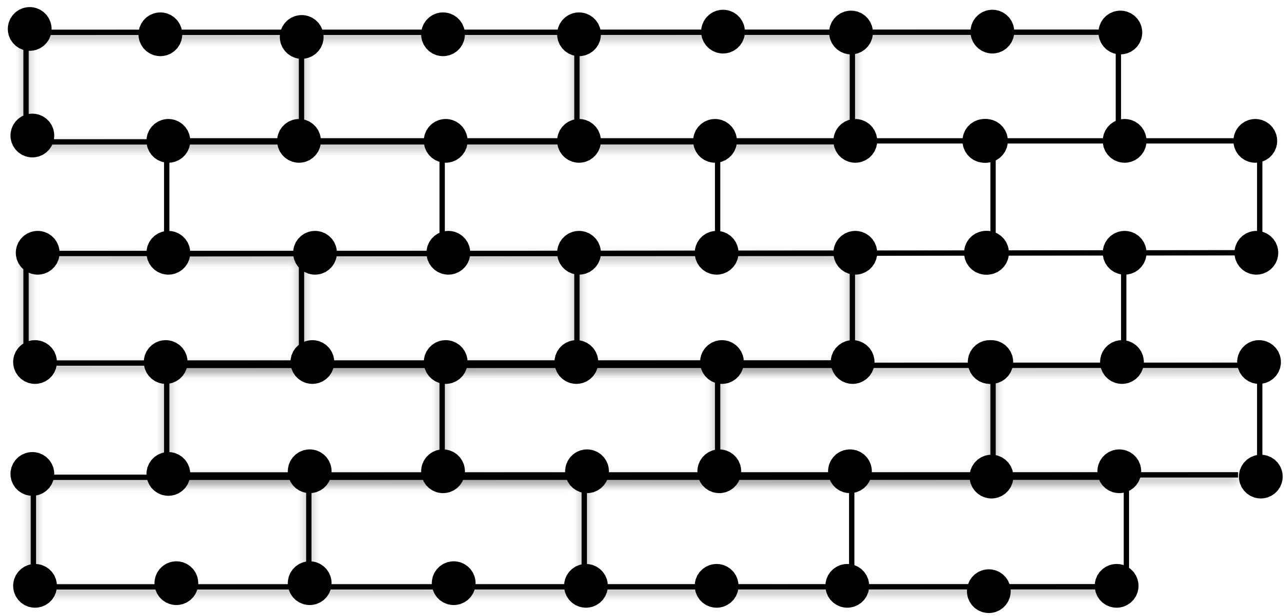
[C, Kim, Nimavat '16]

$2^{\Omega(\sqrt{\log n})}$-hardness of approximation for subgraphs of grids

Approximation Status of EDP

- $O(\sqrt{n})$-approximation algorithm [Chekuri, Khanna, Shepherd '06]
- $2^{\Omega(\sqrt{\log n})}$ -hardness of approximation even for subgraphs of wall graphs [C, Kim, Nimavat '16]

A Wall



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- $2^{\Omega(\sqrt{\log n})}$-hardness of approximation even for subgraphs of wall graphs [C, Kim, Nimavat '16]
- Work in progress: almost polynomial hardness for EDP on wall graphs [C, Kim, Nimavat '17]

Summary so Far

EDP and NDP do not have reasonable approximation algorithms, even on planar graphs

What if we allow some congestion?

EDP/NDP with Congestion

An α-approximation algorithm with congestion c routes OPT/α demand pairs with congestion at most c.

up to c paths can share an edge or a vertex

EDP/NDP with Congestion

An α-approximation algorithm with congestion c routes OPT/α demand pairs with congestion at most c.

optimum number of pairs with no congestion allowed

EDP with Congestion

- Congestion O(log n/log log n): constant approximation [Raghavan, Thompson '87]
- Congestion c: $O(n^{1/c})$-approximation [Azar, Regev '01], [Baveja, Srinivasan '00], [Kolliopoulos, Stein '04]
- Congestion poly(log log n): polylog(n)-approx [Andrews '10]
- Congestion 2: $O(n^{3/7})$-approximation [Kawarabayashi, Kobayashi '11]
- Congestion 14: polylog(k)-approximation [C, '11]
- Congestion 2: polylog(k)-approximation [C, Li '12]
- polylog(k)-approximation for NDP with congestion 2 [Chekuri, Ene '12], [Chekuri, C '16]

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All these results are based on the multicommodity flow relaxation

“Tight” due to known hardness results

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- Congestion 2: $O(n^{3/7})$- … rabayashi, Kobayashi '11]

Structural results about graphs

- Congestion 14: polylog(k)-approximation [C, '11]
- … ylog(k)-approximation [C, Li '12]
- … imation for NDP with congestion 2 [Chekuri, Ene '12], [Chekuri, C '16]

new results in graph theory!

EDP with Congestion

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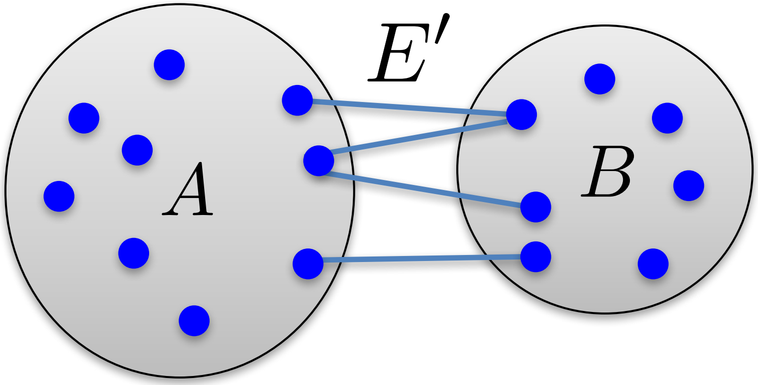
Edge-Disjoint Paths with Constant Congestion

EDP on Expanders

$$|E'| \geq \frac{\min\{|A|, |B|\}}{2}$$



In a strong enough expander, if the set of demand pairs is not too large, can route almost all of them on Node-Disjoint Paths!

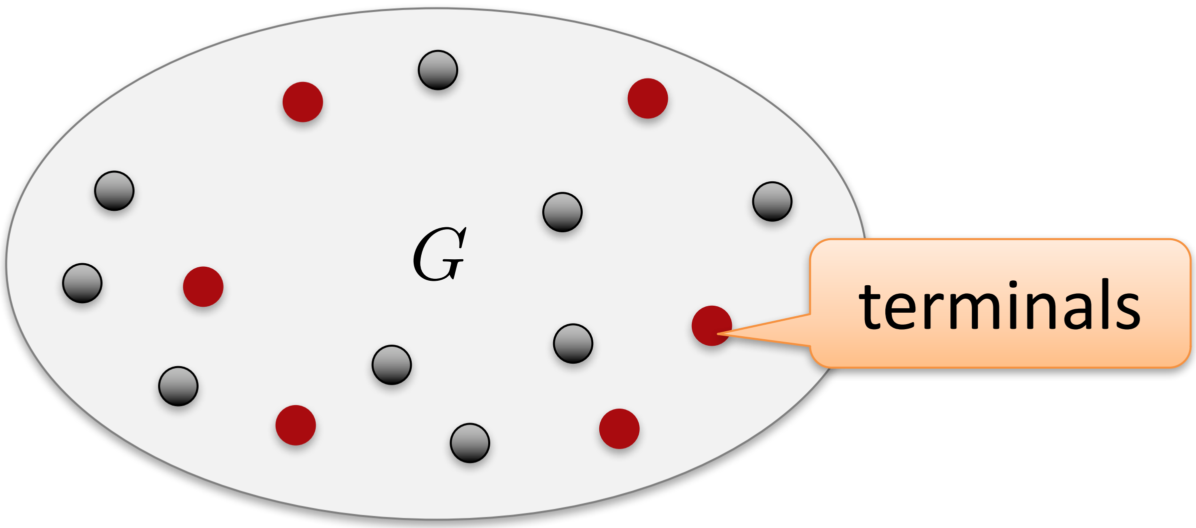
Main Idea: Exploit Algorithms for Expanders!

But our graph is nothing like an expander

Find expander-like structure in the graph and use it for routing!

Well-Linkedness

[Robertson,Seymour], [Chekuri, Khanna, Shepherd], [Raecke]



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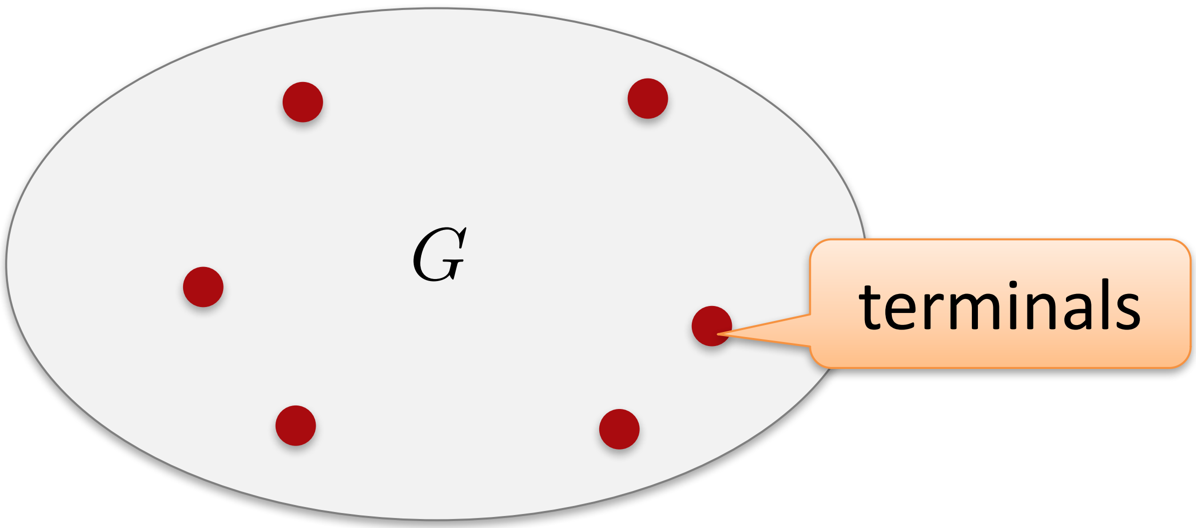
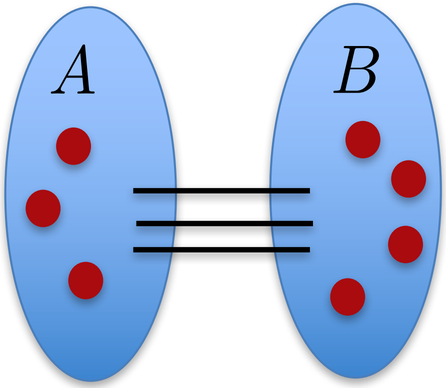


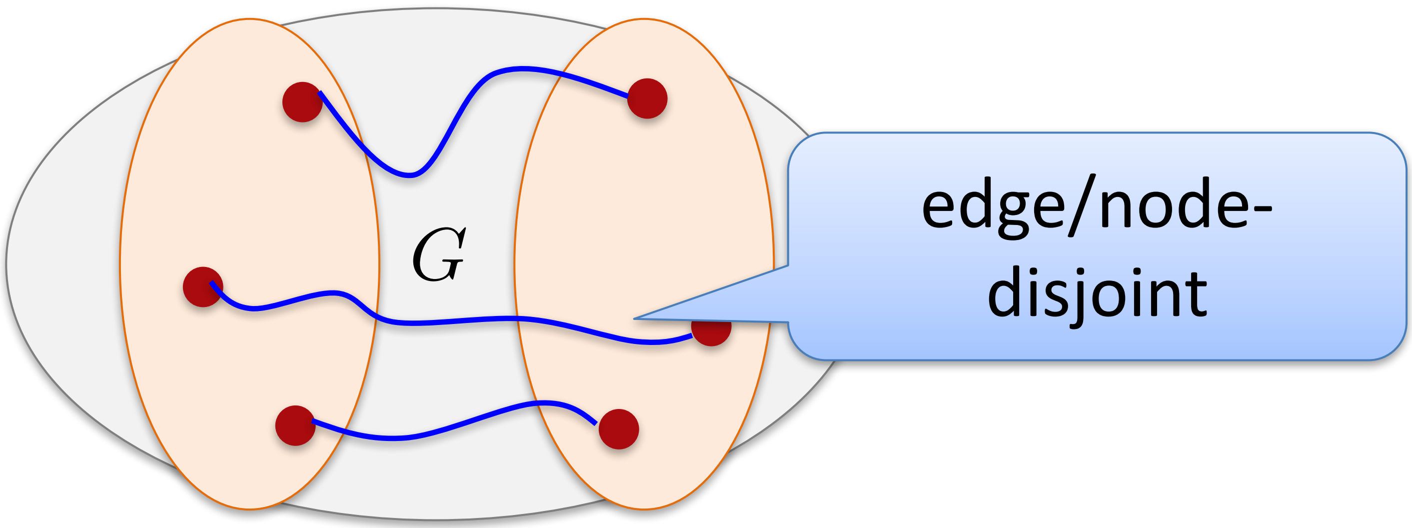
Set T of terminals is well-linked in G, iff for any partition (A,B) of V(G),

$$|E(A,B)| \geq \min\{|A \cap T|, |B \cap T|\}$$



Well-Linkedness

[Robertson,Seymour], [Chekuri, Khanna, Shepherd], [Raecke]



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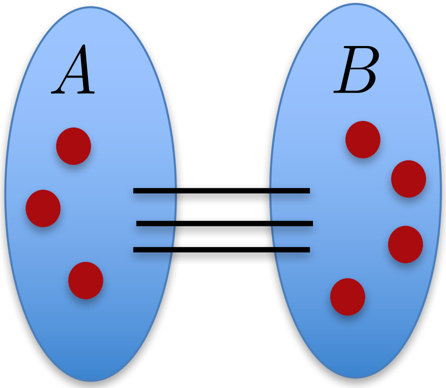


EDP: Well-Linked Instances

- Terminals: vertices participating in the demand pairs

- An instance is well-linked iff the set of terminals is well-linked in G.

Theorem [Chekuri, Khanna Shepherd '04]: an α-approximation algorithm on well-linked instances gives an $O(\alpha \log^2 k)$-approximation on any instance.

EDP: Well-Linked Instances

- Terminals: vertices participating in the demand pairs

- An instance is well-linked iff t
terminals is well-linked in G.

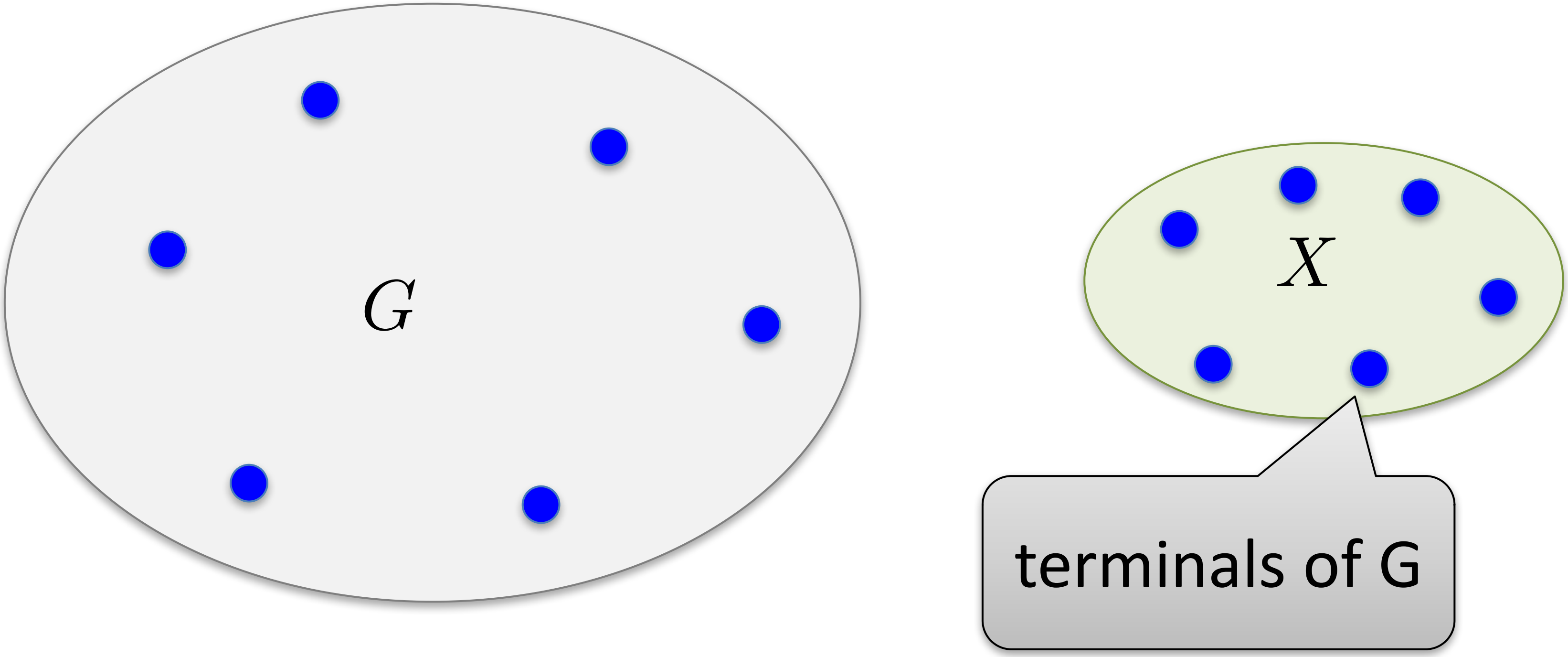
Only true if the algorithm rounds the flow relaxation

Theorem [Chekuri, Khanna Shepherd '04]: an α-approximation algorithm on well-linked instances gives an $O(\alpha \log^2 k)$-approximation on any instance.

Main Idea

[Chekuri, Khanna, Shepherd], [Rao, Zhou]

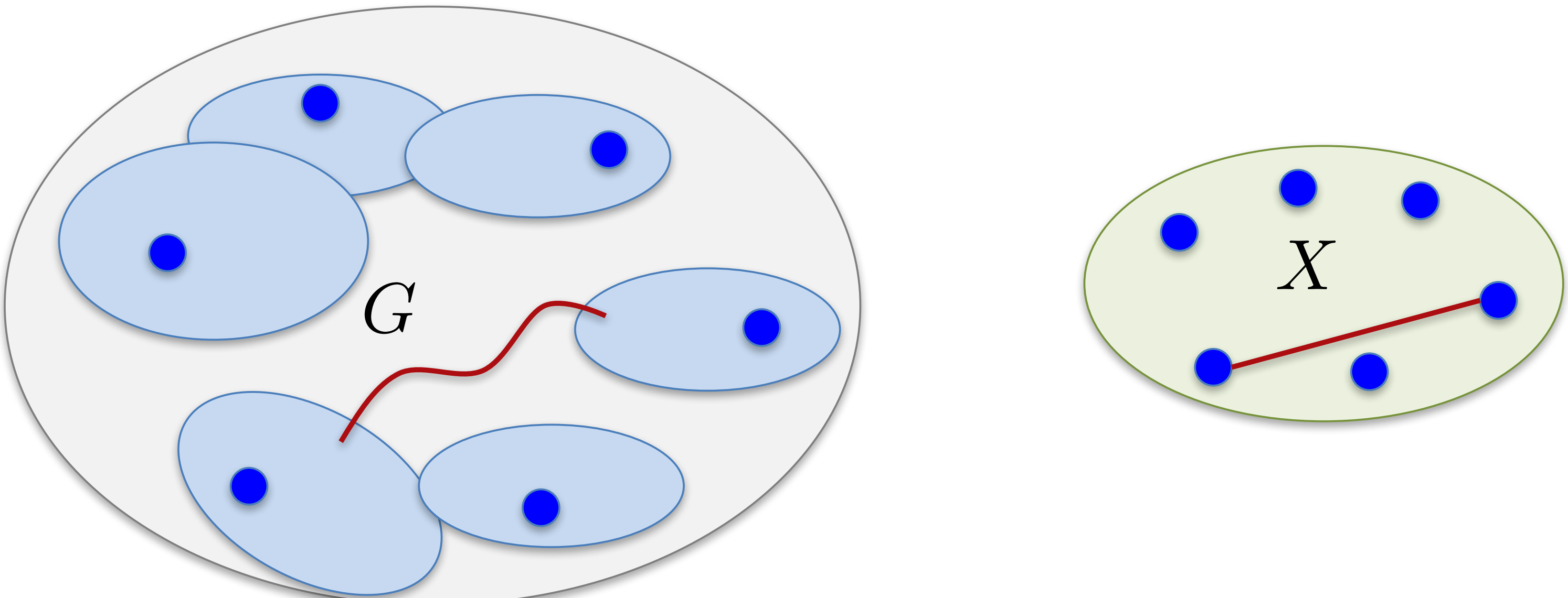
Embed an expander over the terminals into G!



Main Idea

[Chekuri, Khanna, Shepherd], [Rao, Zhou]

Embed an expander over the terminals into G!



An edge of G may belong to at most 2 clusters/paths

Main Idea

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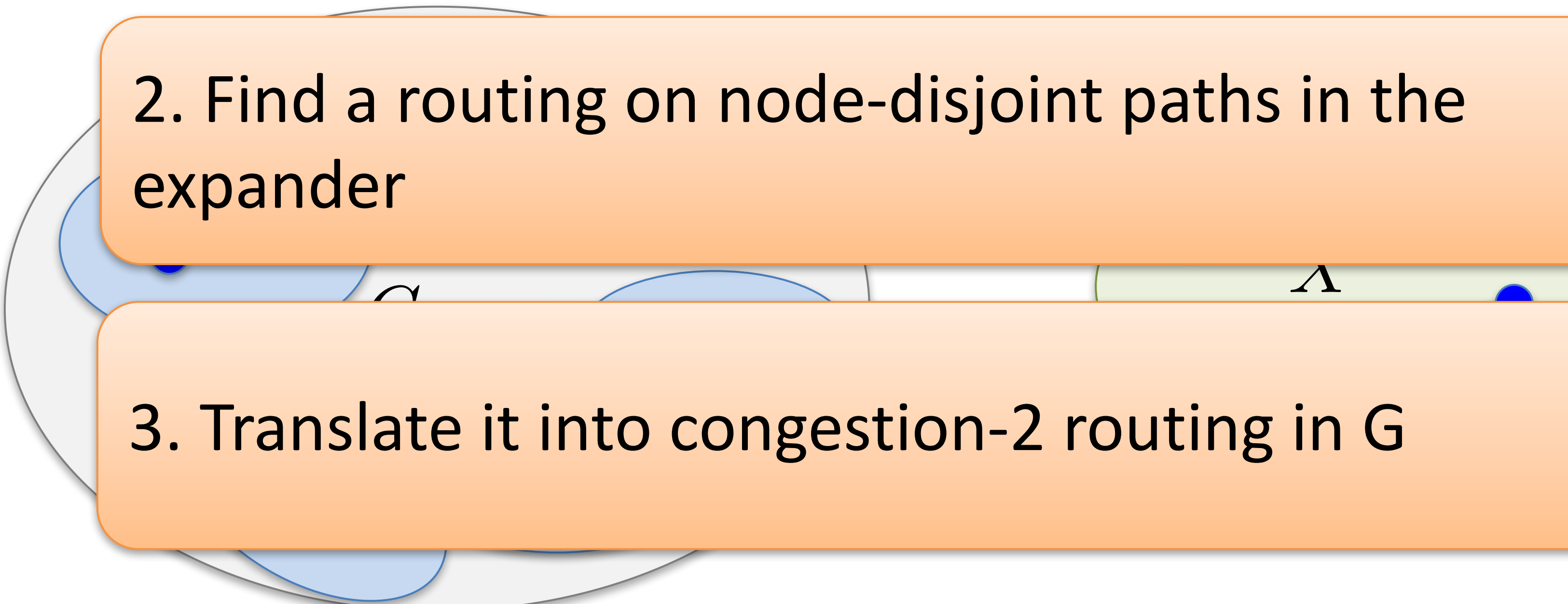
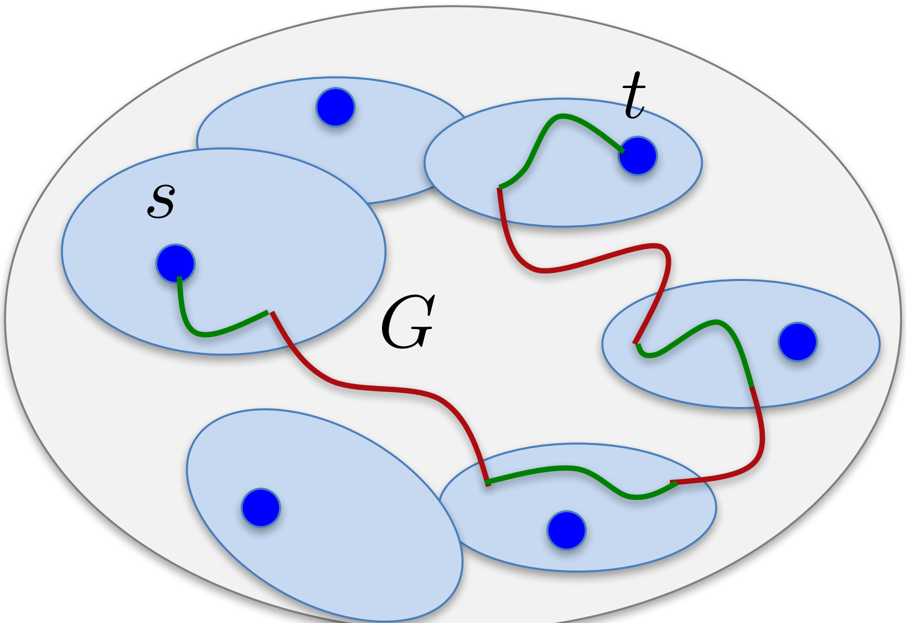


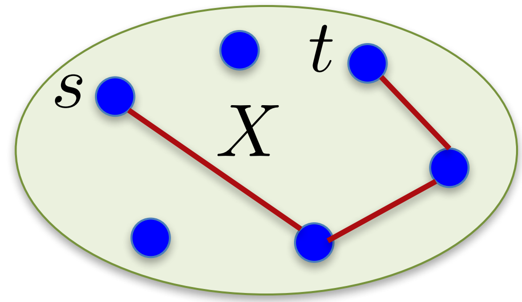
2. Find a routing on node-disjoint paths in the expander

3. Translate it into congestion-2 routing in G

An edge of G may belong to at most 2 clusters/paths

Embedding an Expander into G





Routing on vertex-disjoint paths in X gives a good routing in G!

Main Idea

1. Embed an expander over the terminals into G

2. Find a routing on node-disjoint paths in the expander

3. Translate it into congestion-2 routing in G

An edge of G may belong to at most 2 clusters/paths

Main Idea

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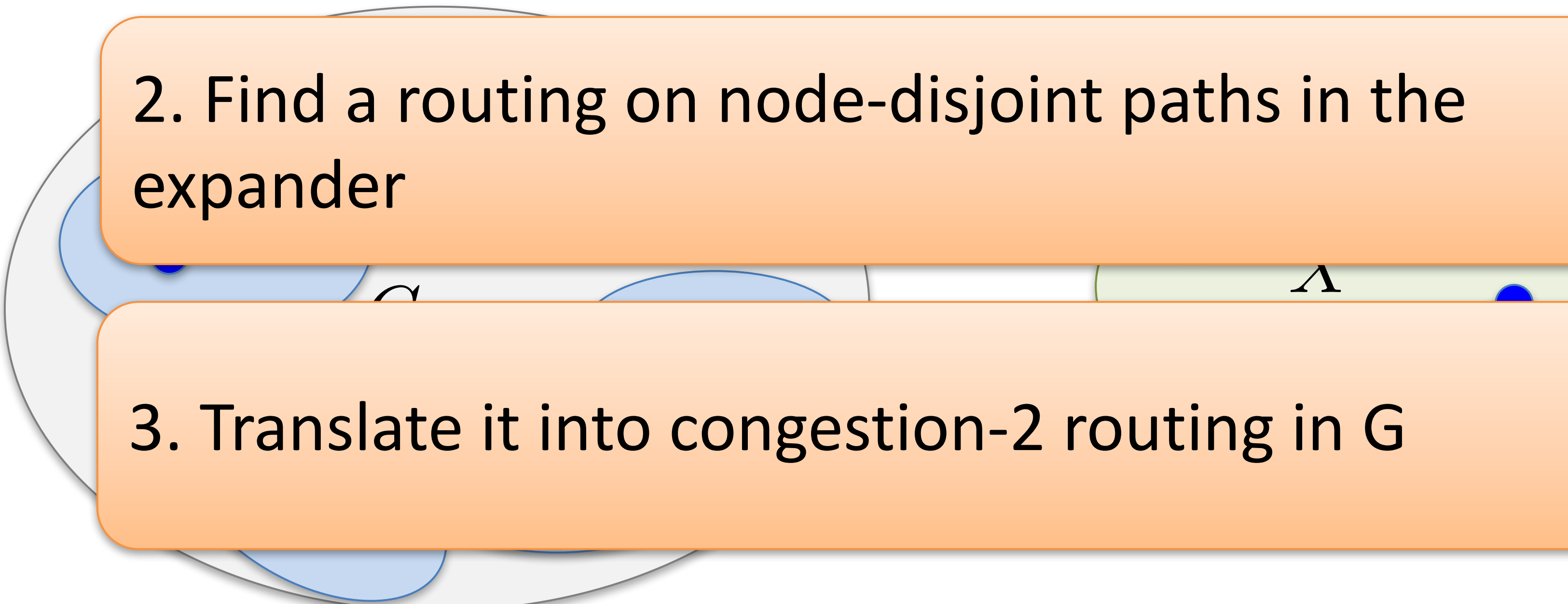

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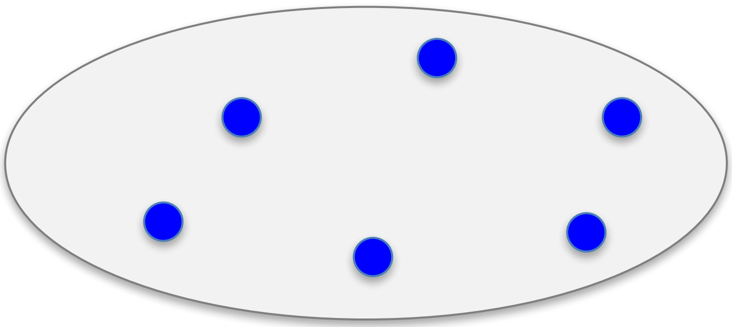
Cut-Matching Game [Khandekar, Rao, Vazirani '06]

Cut Player: wants to build an expander

Matching Player: wants to delay its construction



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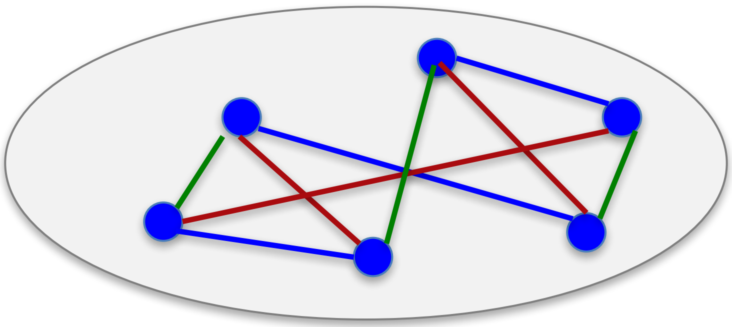
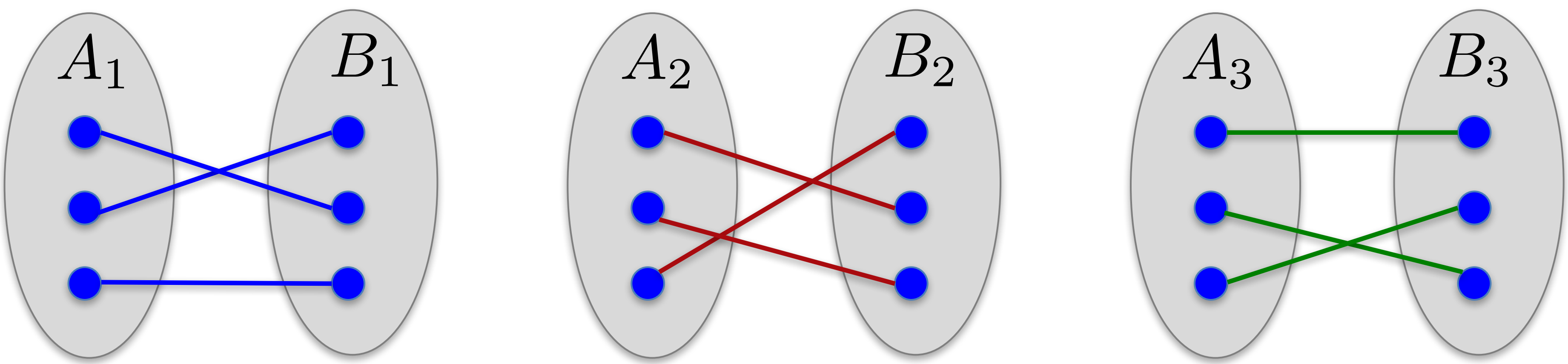
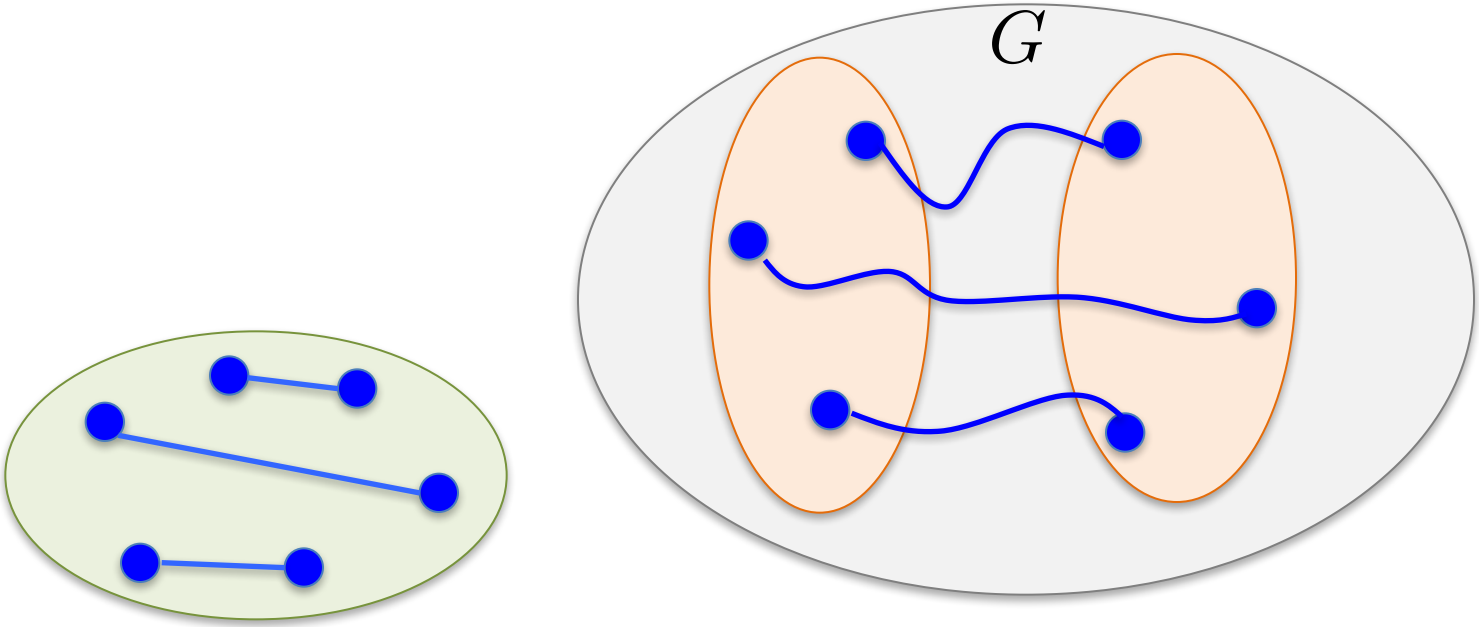


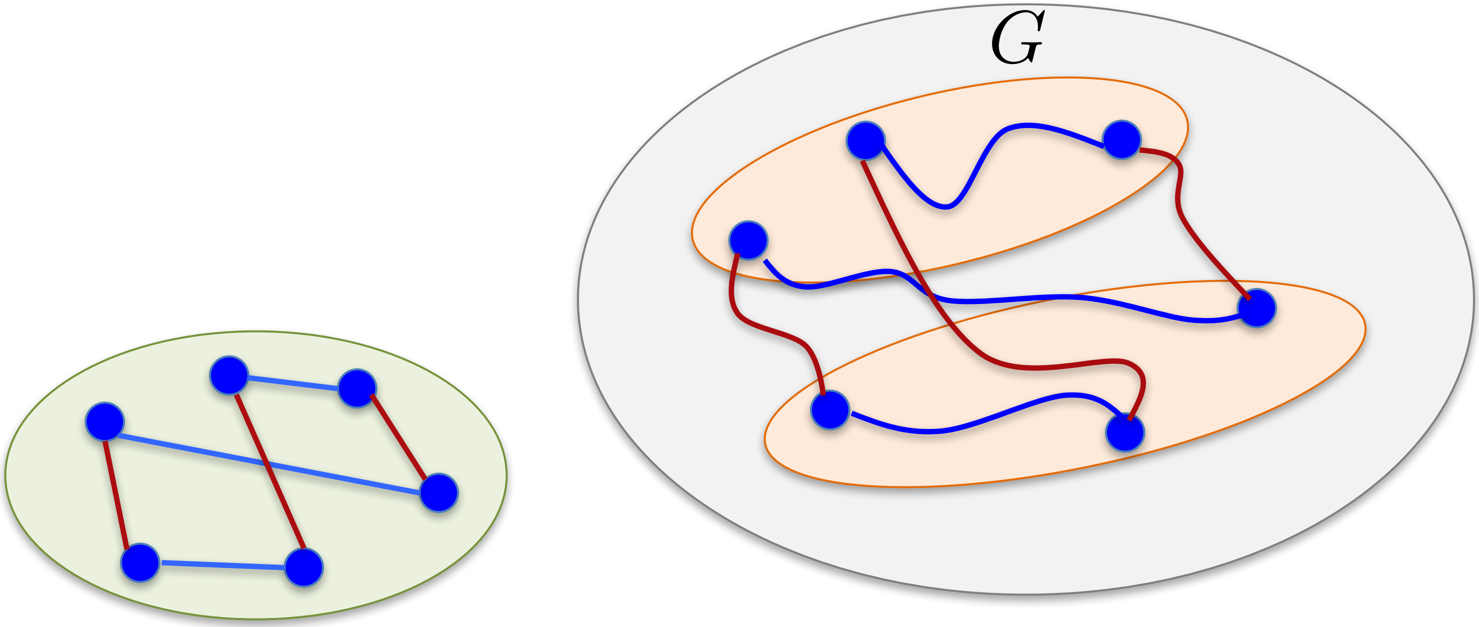
There is a strategy for cut player, s.t. after O(log^{2}n) iterations, we get an expander!



Embedding Expander into Graph



Embedding Expander into Graph



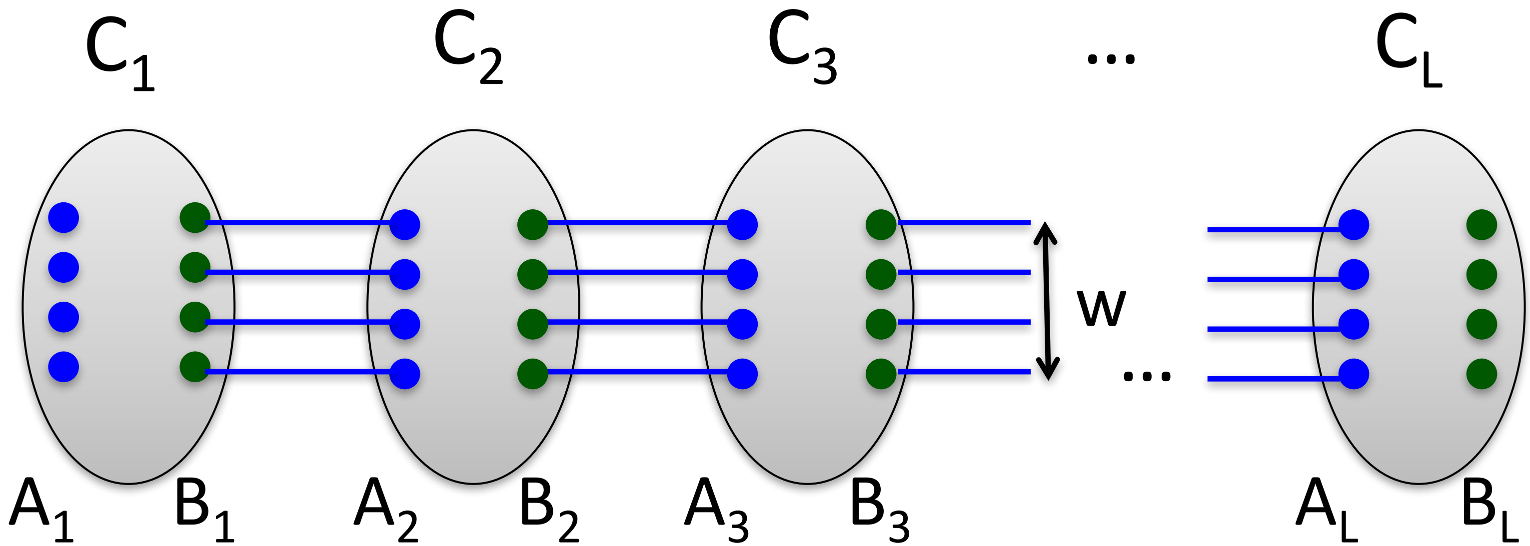
After $O(\log^2 k)$ iterations, we get an expander embedded into G.

Problem: congestion $\Omega(\log^2 k)$

Path-of-Sets System

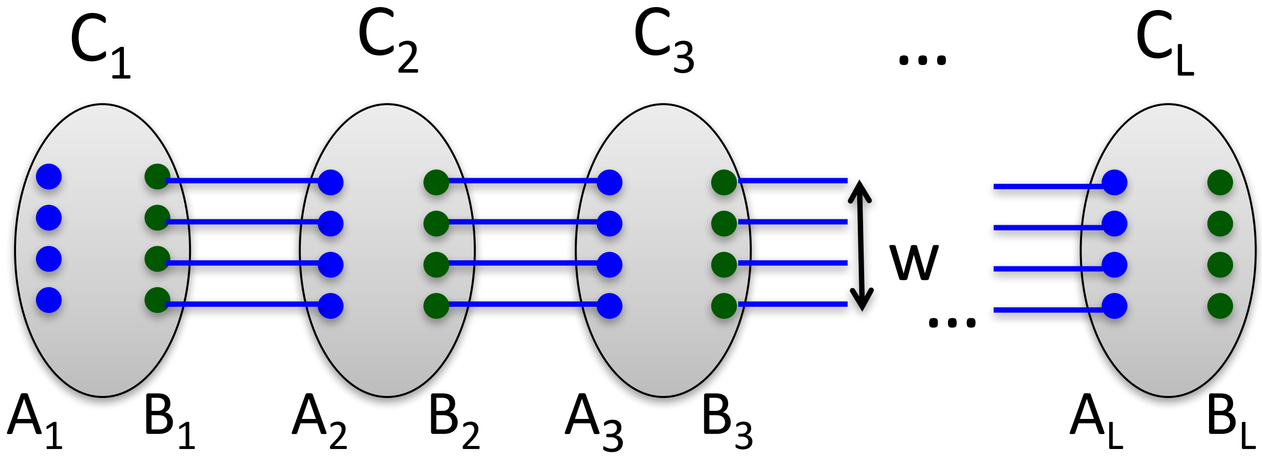
A Path-of-Sets System

width w
length L



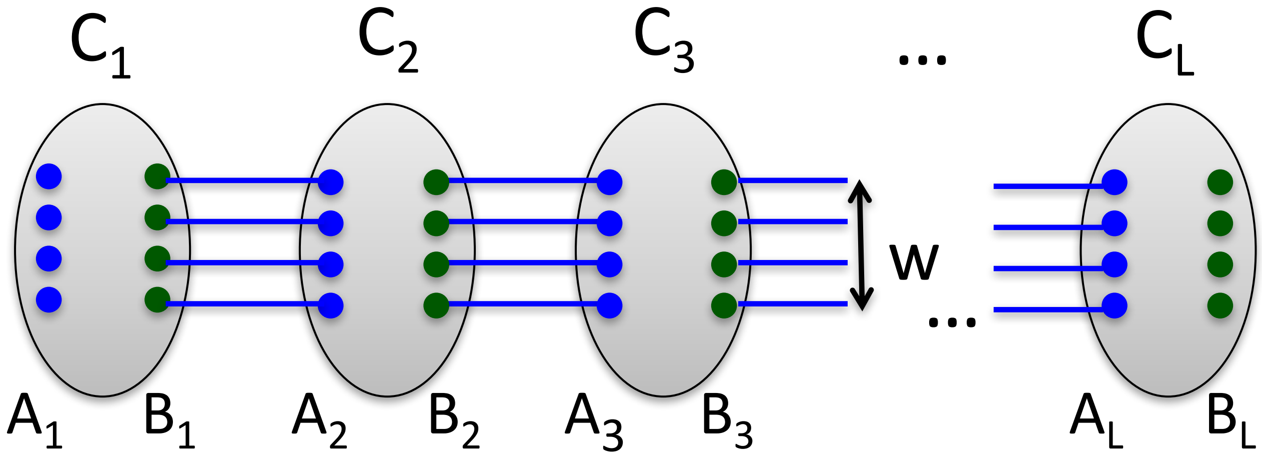
- L disjoint connected clusters
- Two disjoint sets A_i, B_i of w vertices in each cluster C_i
- $A_i \cup B_i$ is well-linked in C_i
- For all i, set P_i of w disjoint paths connecting B_i to A_{i+1}
- All paths are disjoint from each other and internally disjoint from clusters

From Well-Linkedness to Path-of-Sets



Theorem [C, '11], [C, Li '12], [Chekuri, C '13]:
Suppose G has a set of k well-linked vertices.
Then we can efficiently construct a path-of-sets system in G with parameters L and w, if: $w \cdot L^{48} < \tilde{O}(k)$

From Well-Linkedness to Path-of-Sets



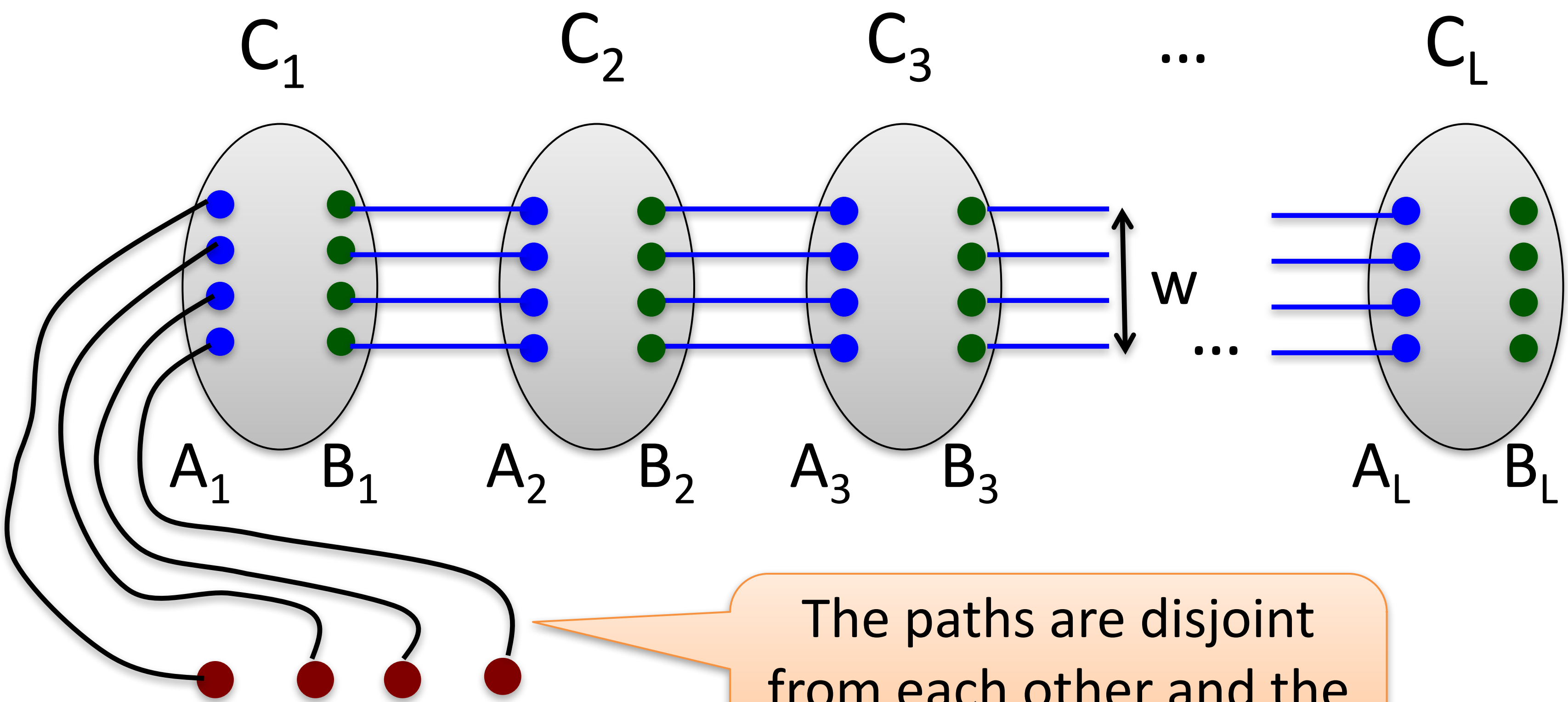
We'll use:
$L=O(\log^2 k)$
$w=k/\text{polylog } k$

Theorem [C, '11], [C, Li '12], [Chekuri, C '13]:
Suppose G has a set of k well-linked vertices.
Then we can efficiently construct a path-of-sets system in G with parameters L and w, if: $w \cdot L^{48} < \tilde{O}(k)$

Extras:

- Can connect w terminals to A_1 by disjoint paths
- Can make sure they form demand pairs!

From Well-Linkedness to Path-of-Sets

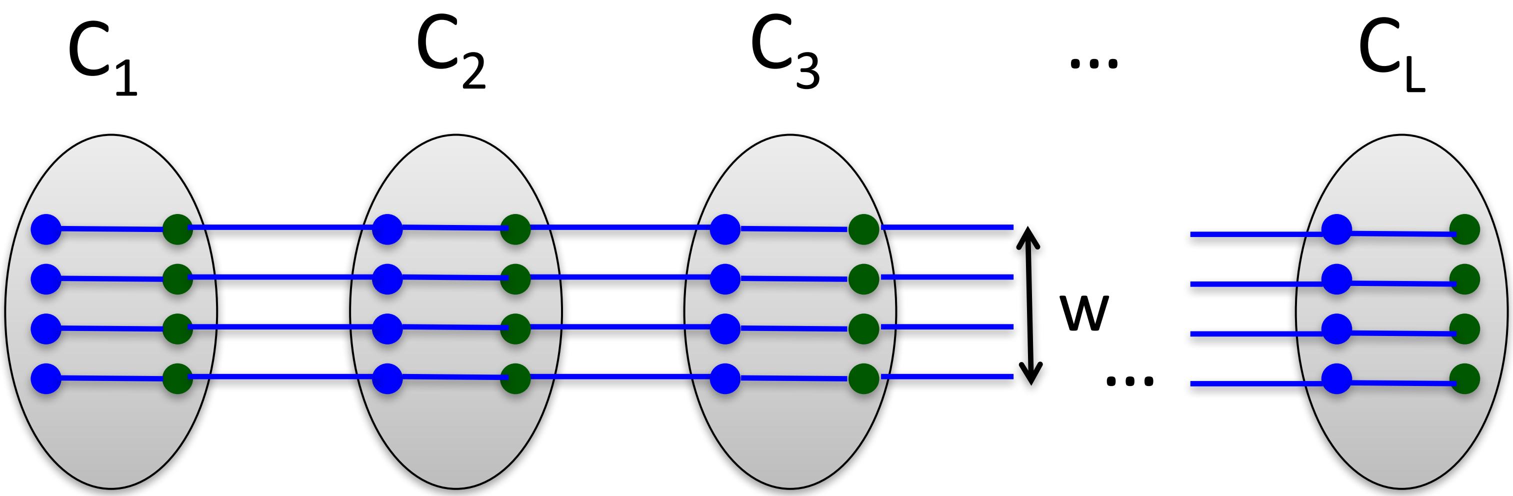


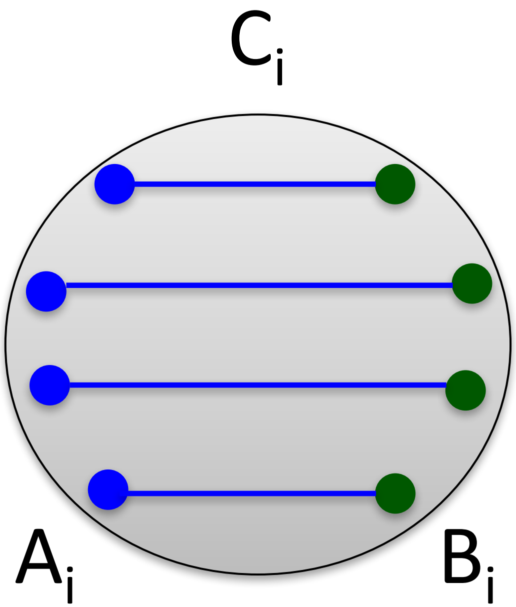
The paths are disjoint from each other and the PoS system

The terminals form demand pairs

Given the PoS, can embed an expander!

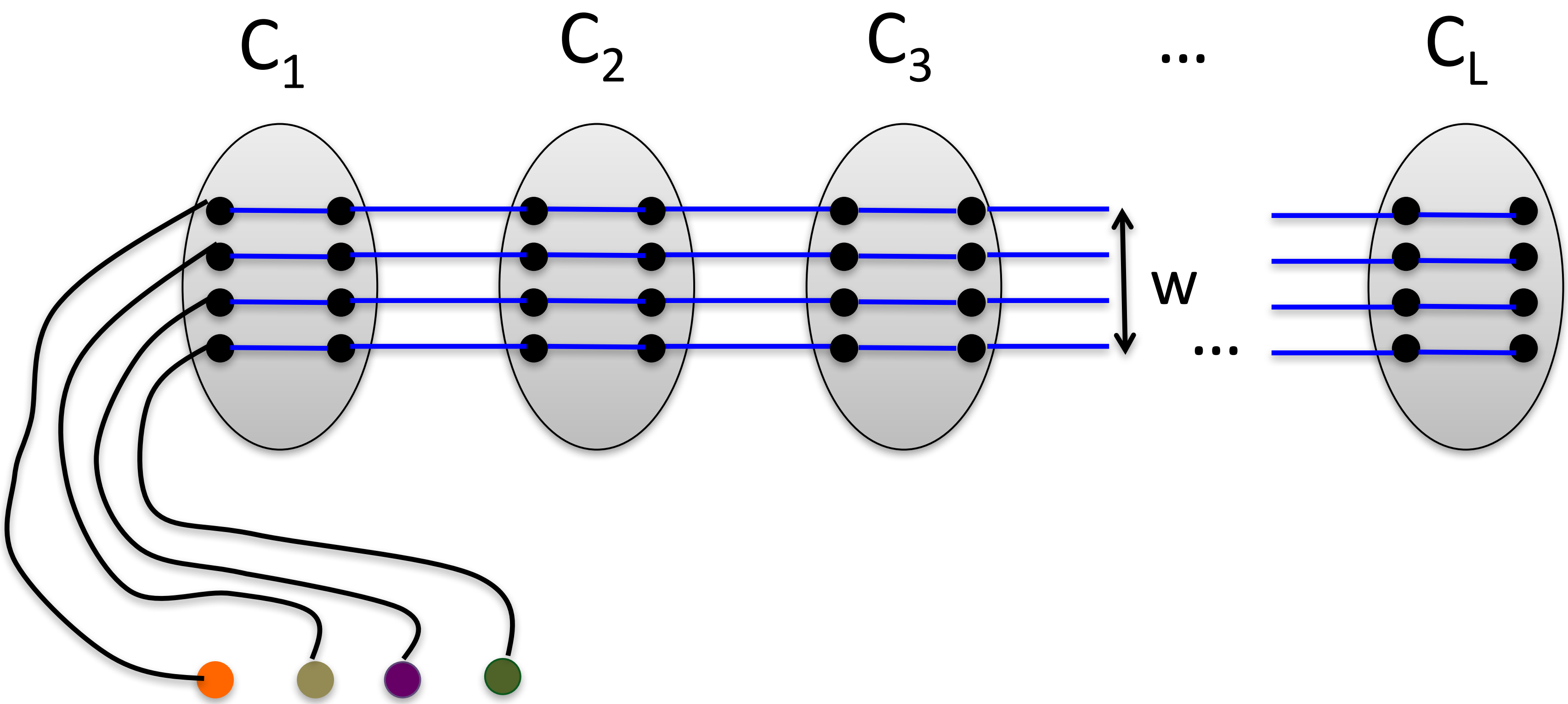
Embedding the Expander

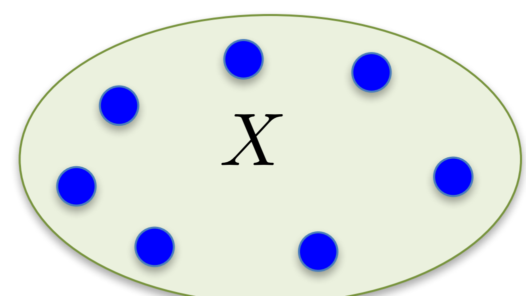




$A_i \cup B_i$ is well-linked inside C_i

Embedding the Expander





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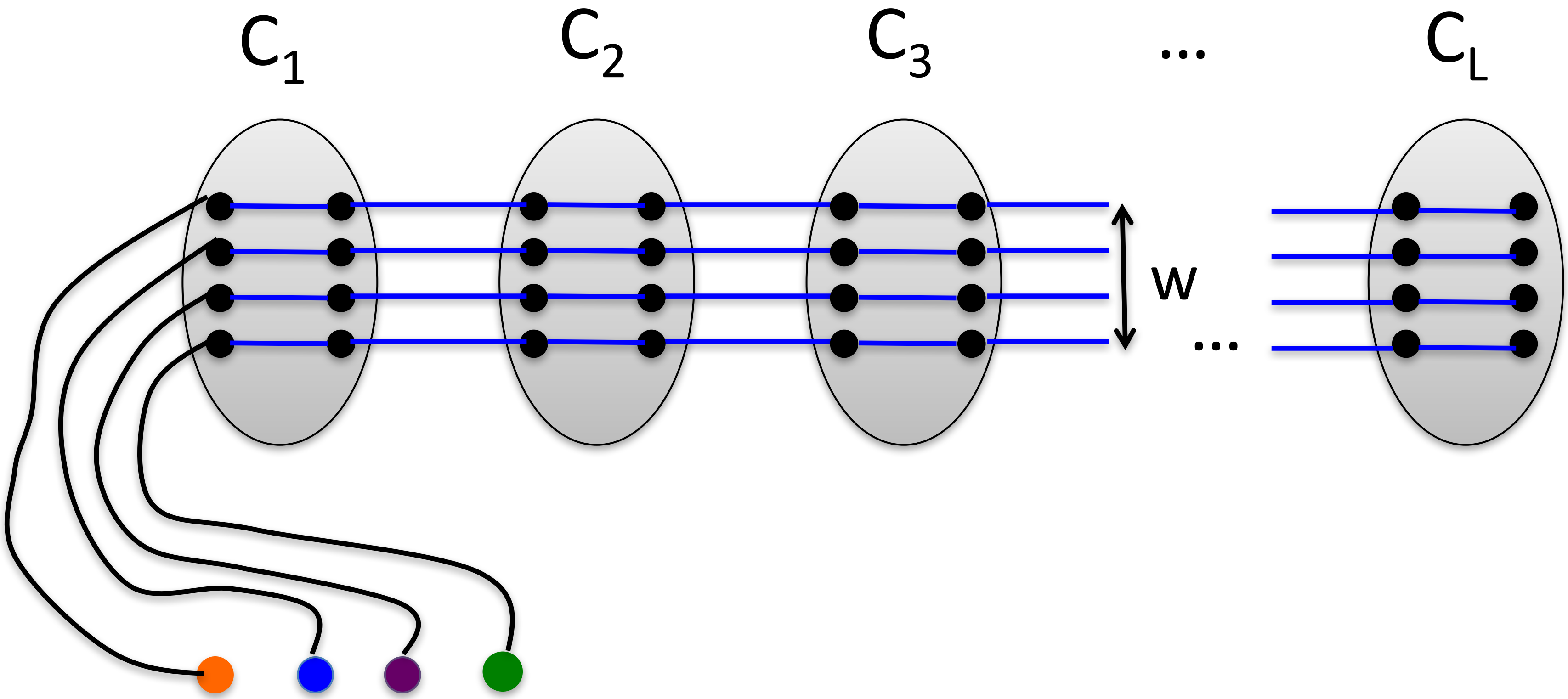
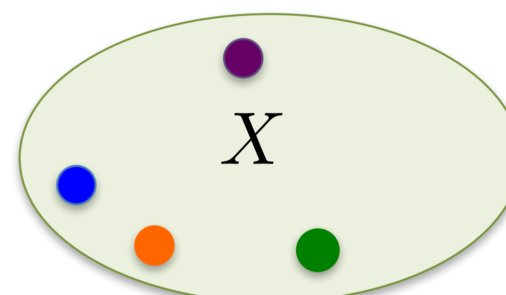


Expander vertex → the path containing the terminal



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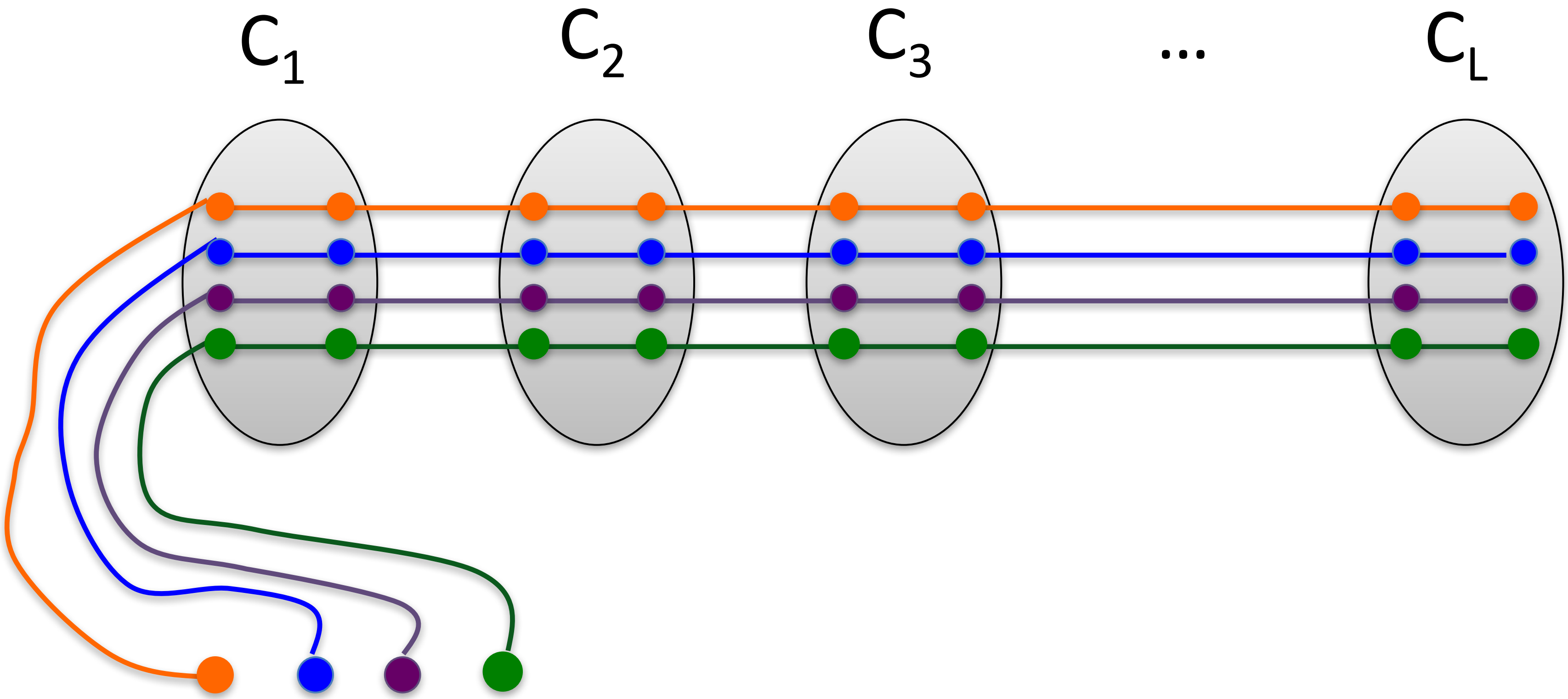

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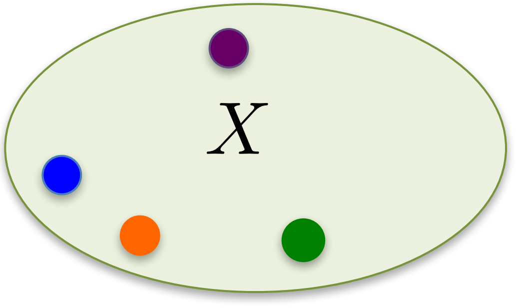

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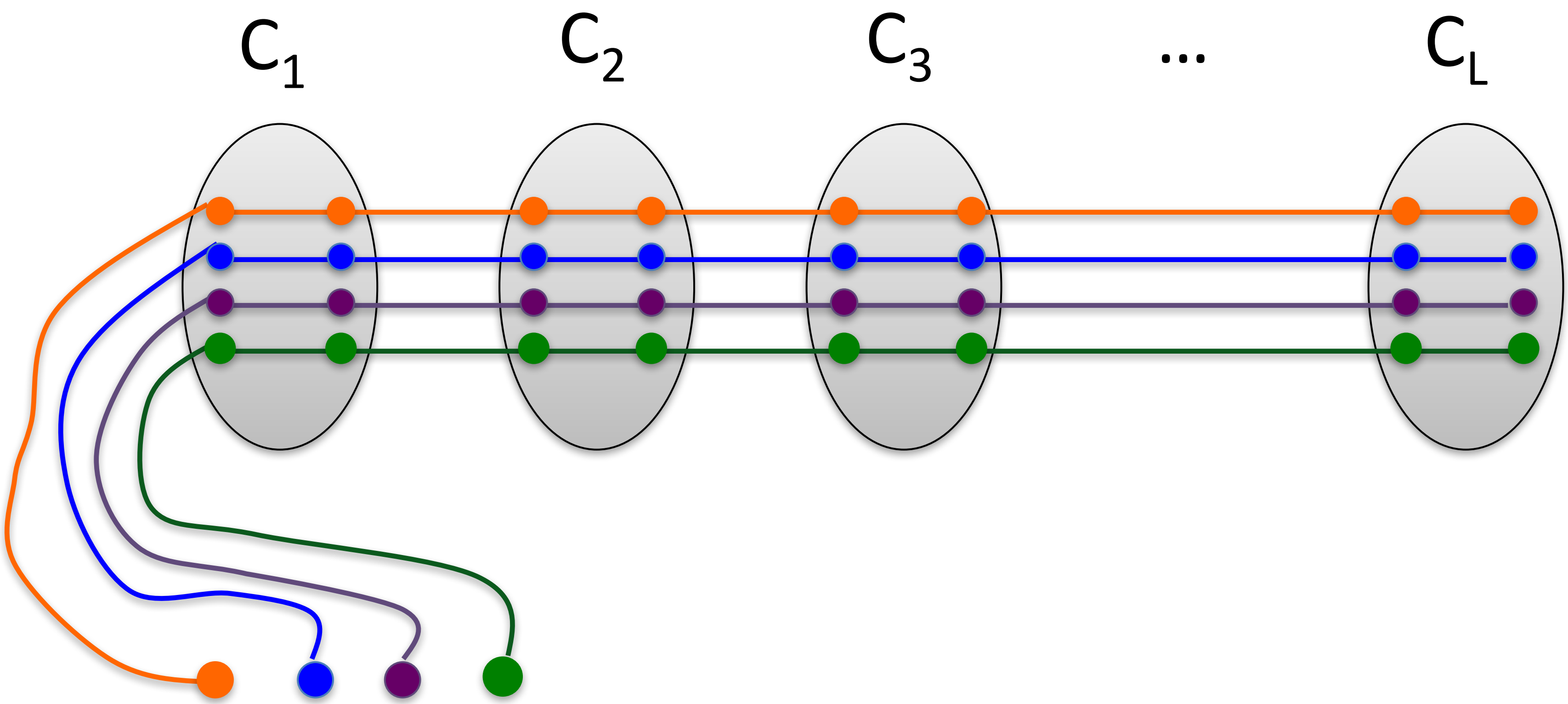
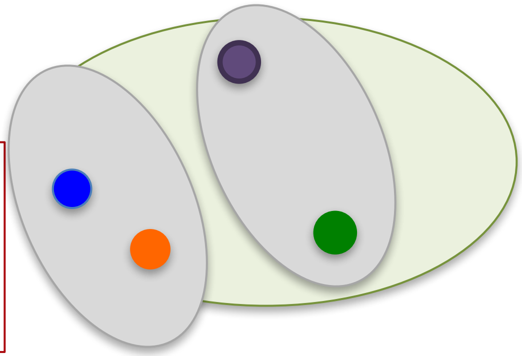


Expander edges? → cut-matching game!



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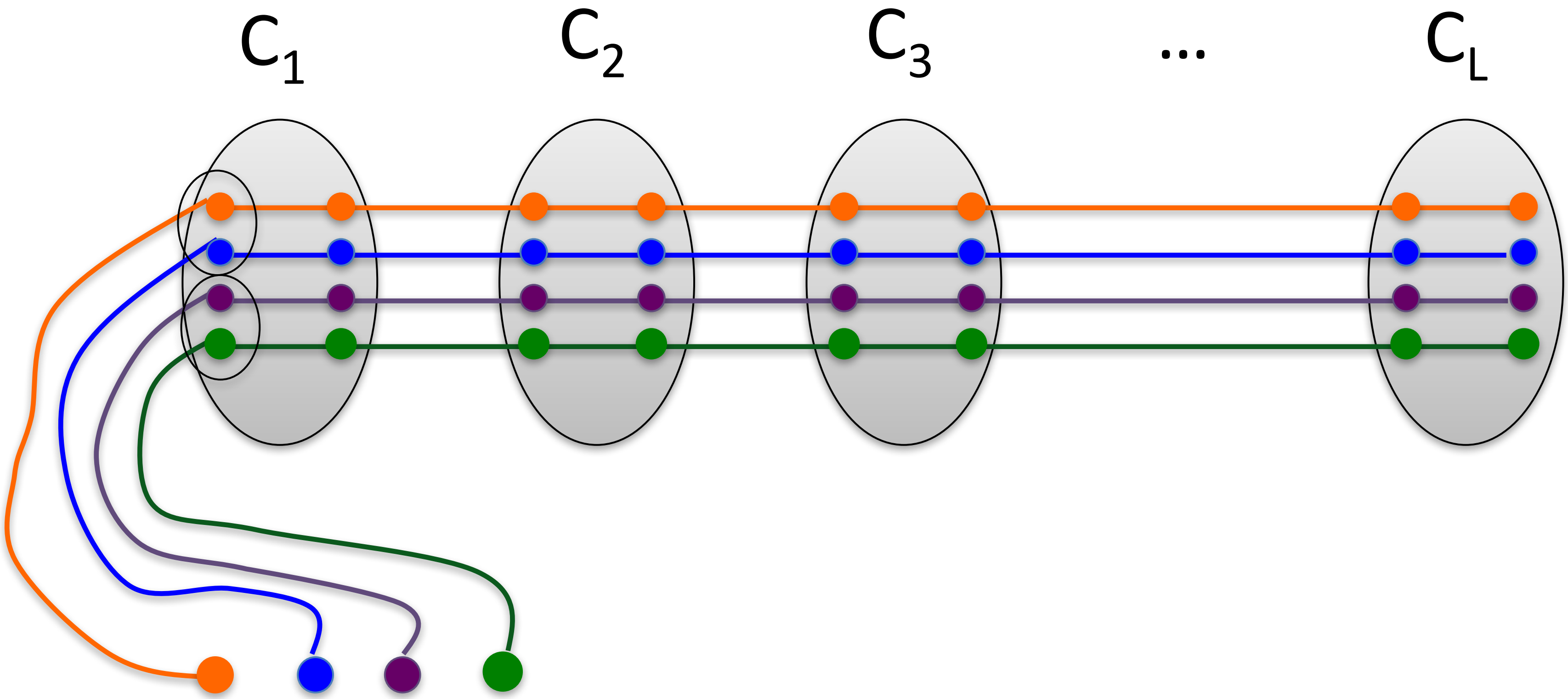

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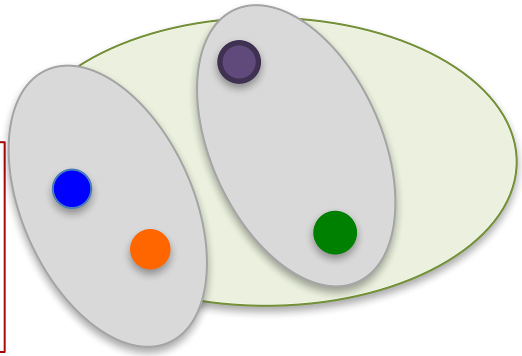

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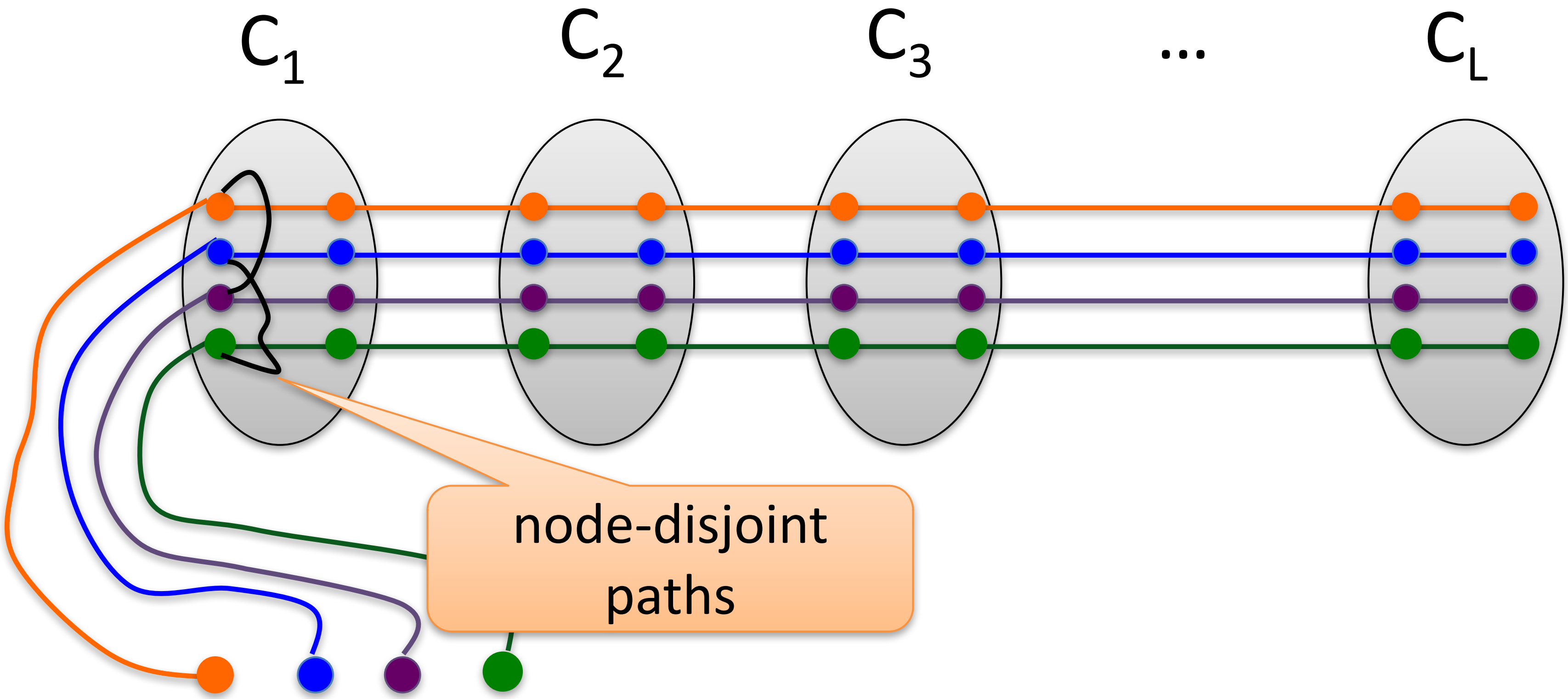

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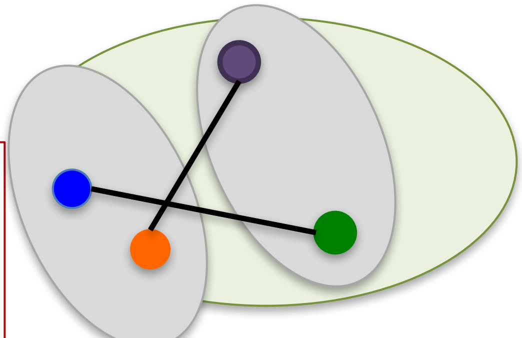

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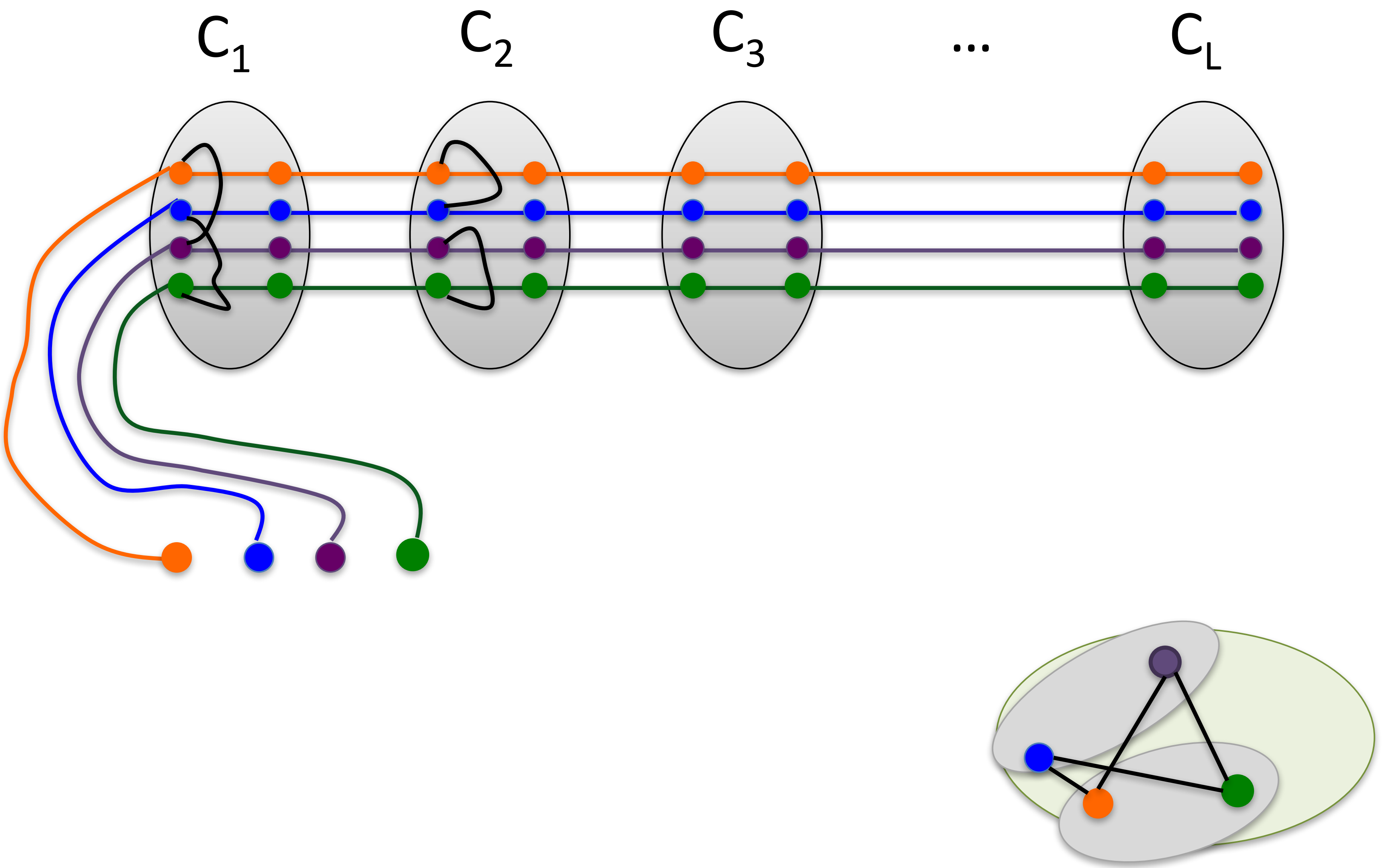

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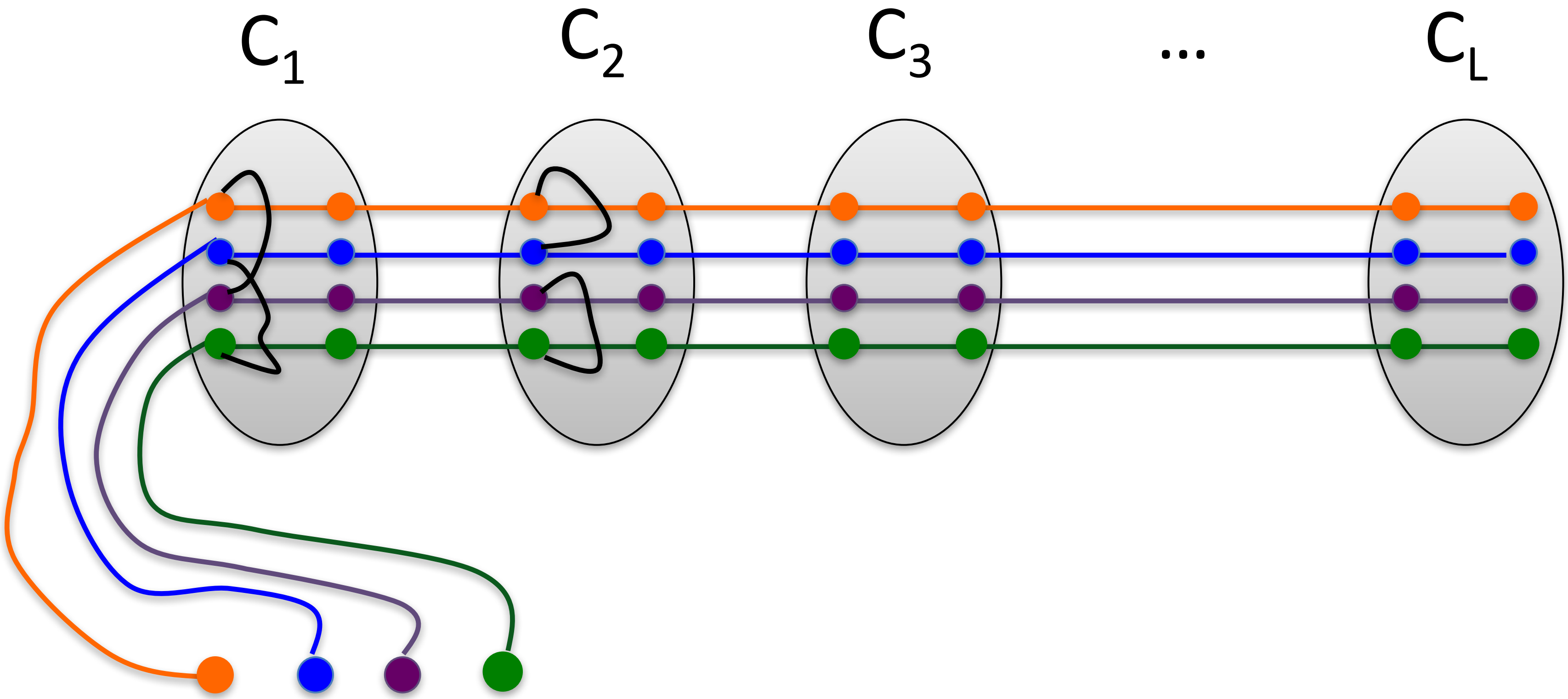
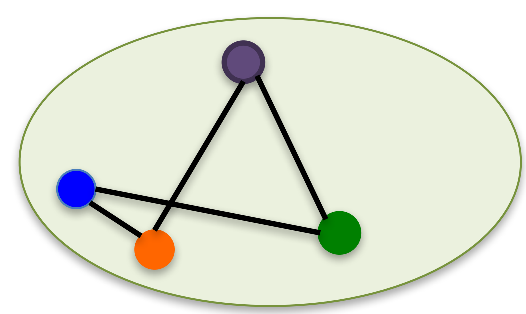


After $O(\log^2 k)$ iterations, we obtain an expander embedded into G with congestion 2.



Algorithm for EDPwC in Well-Linked Instances

Find a Path-of-Sets System

Embed an expander into G

Find vertex-disjoint routing in the expander

Transform into routing in G

Structural Result

If G contains a large well-linked set of vertices, then it contains a large Path-of-Sets System

Treewidth sparsifiers

Excluded grid theorem

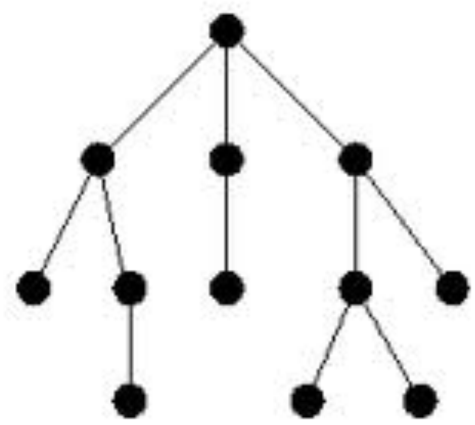
Large-treewidth graph decompositions

Vertex flow sparsifiers

Excluded Grid Theorem [Robertson, Seymour]

Excluded Grid Theorem [Robertson, Seymour]

Simple graphs



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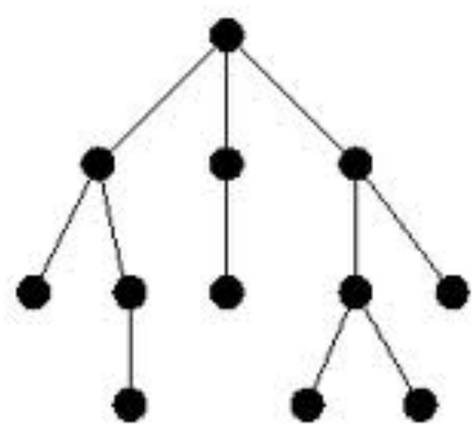
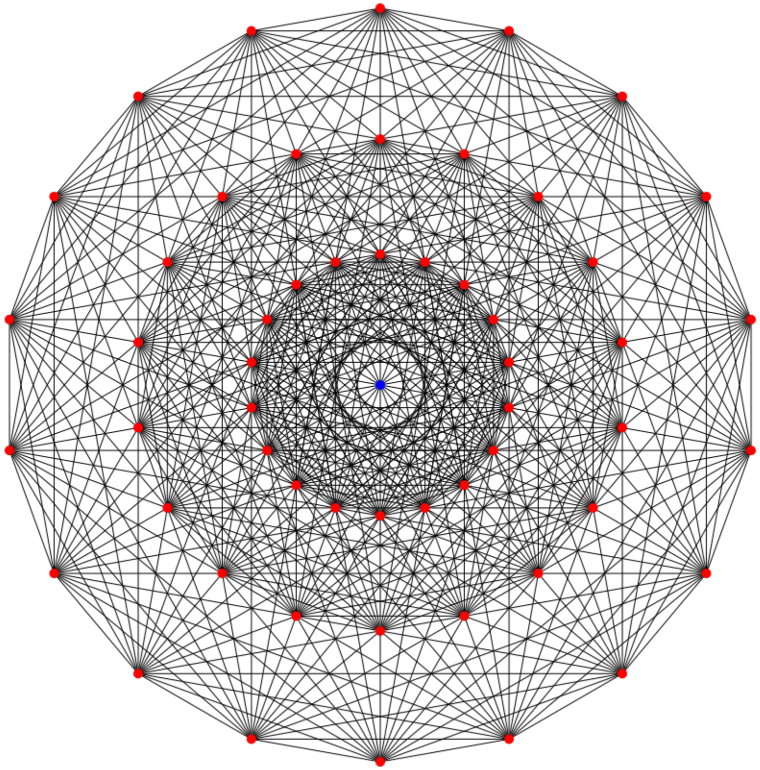


Complicated graphs

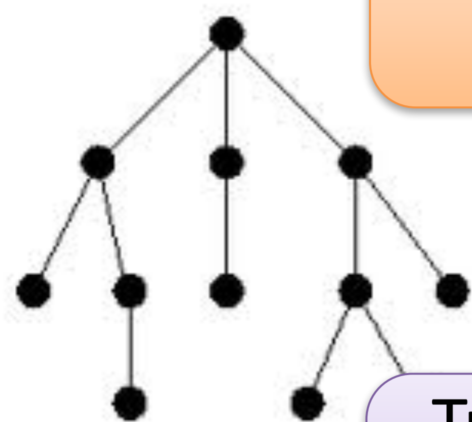


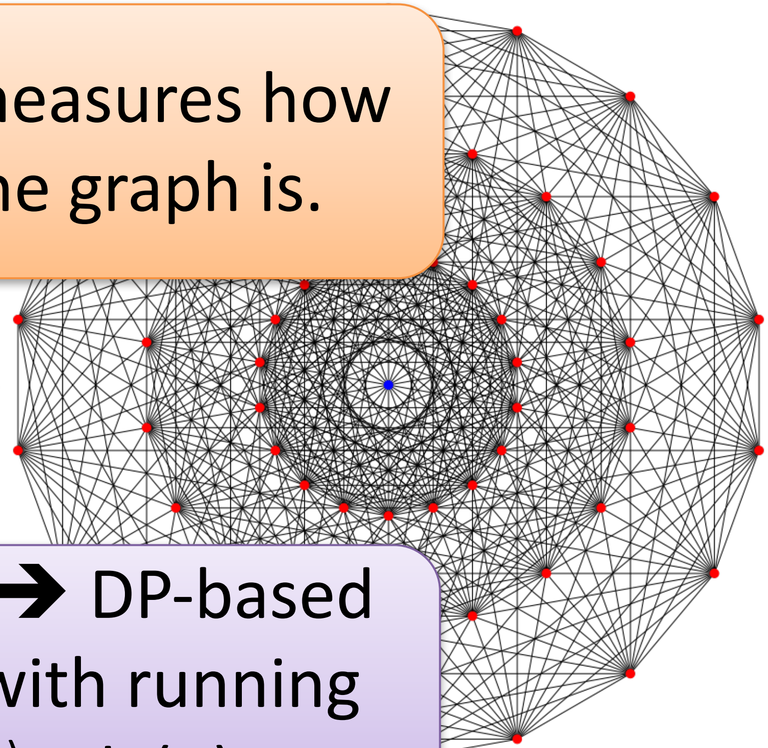
Excluded Grid Theorem [Robertson, Seymour]

Simple graphs

Complicated graphs

Treewidth: measures how complex the graph is.



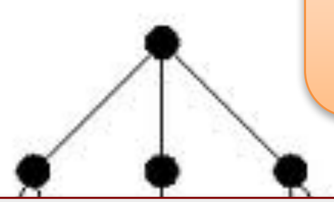


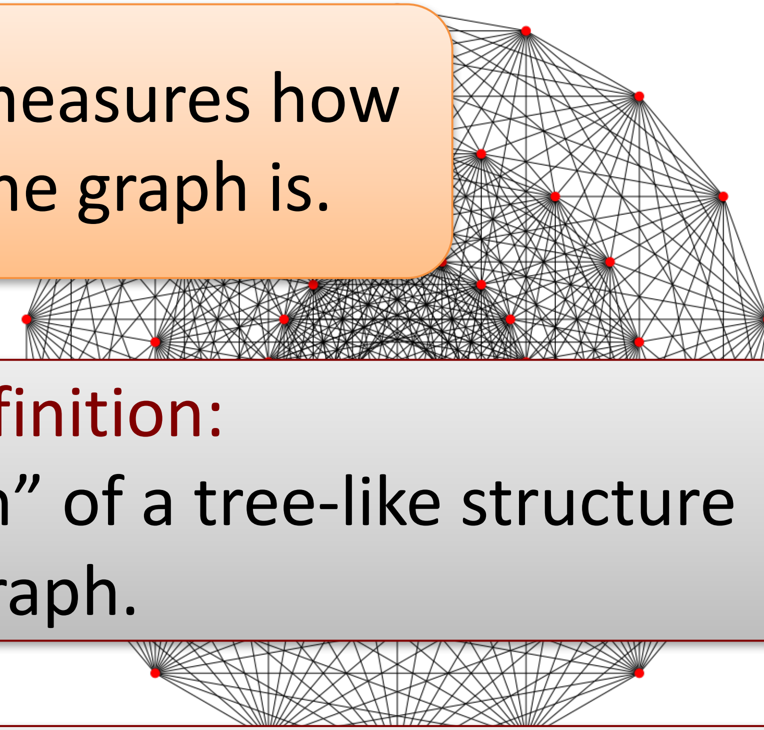
Treewidth k → DP-based algorithms with running time $2^{O(k)}$poly(n).

Excluded Grid Theorem [Robertson, Seymour]

Simple graphs

Complicated graphs





Treewidth: measures how complex the graph is.

Original definition:

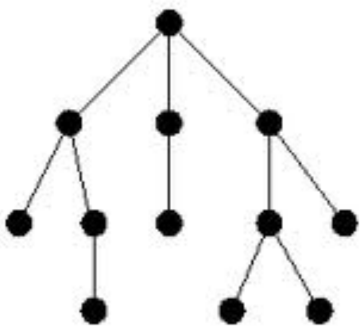
Treewidth is the smallest "width" of a tree-like structure that correctly "simulates" the graph.

(Almost) Equivalent definition:

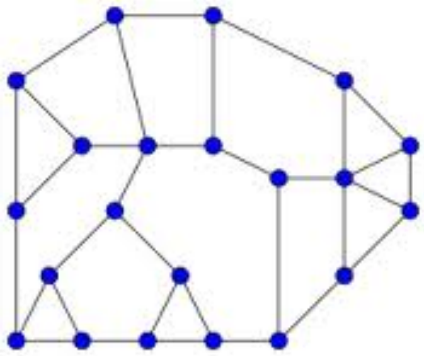
Treewidth is the cardinality of the largest well-linked set of vertices in the graph.

Treewidth

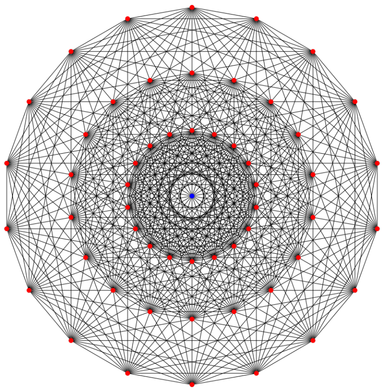
Trees



Low-Treewidth Graphs

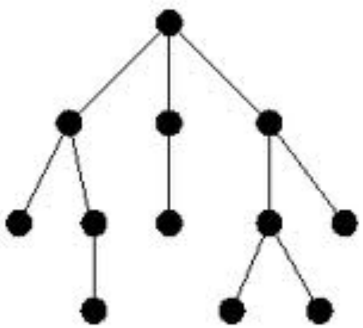


High-Treewidth Graphs



Treewidth

Trees



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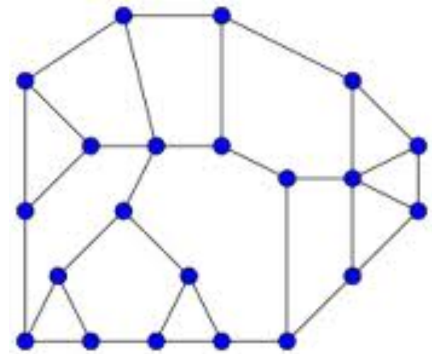

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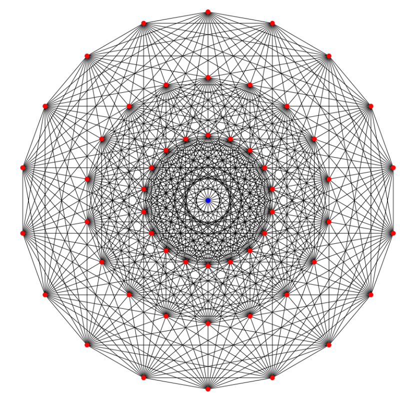


Excluded Grid Theorem

[Robertson, Seymour '86]

If the treewidth of G is large, then G contains a large grid as a minor.

Can embed a large grid into G with no congestion

Excluded Grid Theorem [Robertson, Seymour]

If the treewidth of G is large, then it contains a large grid minor, so:

- G contains many disjoint cycles
- G contains many disjoint cycles of length 0 mod m
- G contains a convenient routing structure
- The size of the vertex cover in G is large
- …

Applications

- Fixed parameter tractability
- Erdos-Posa type results
- Graph minor theory
  - Algorithm for NDP where k is small
- Algorithms for graph crossing number
- …

Excluded Grid Theorem

[Robertson, Seymour ‘86]

If the treewidth of G is k, then G contains a grid of size f(k)xf(k) as a minor.

Excluded Grid Theorem

[Robertson, Seymour '86]

If the treewidth of G is k, then G contains a grid of size f(k)xf(k) as a minor.

How large is f(k)?

- [Robertson, Seymour '94]: $$f(k) = O\left(\sqrt{k/\log k}\right)$$
- Conjecture [Robertson, Seymour '94]: This is tight.

Excluded Grid Theorem

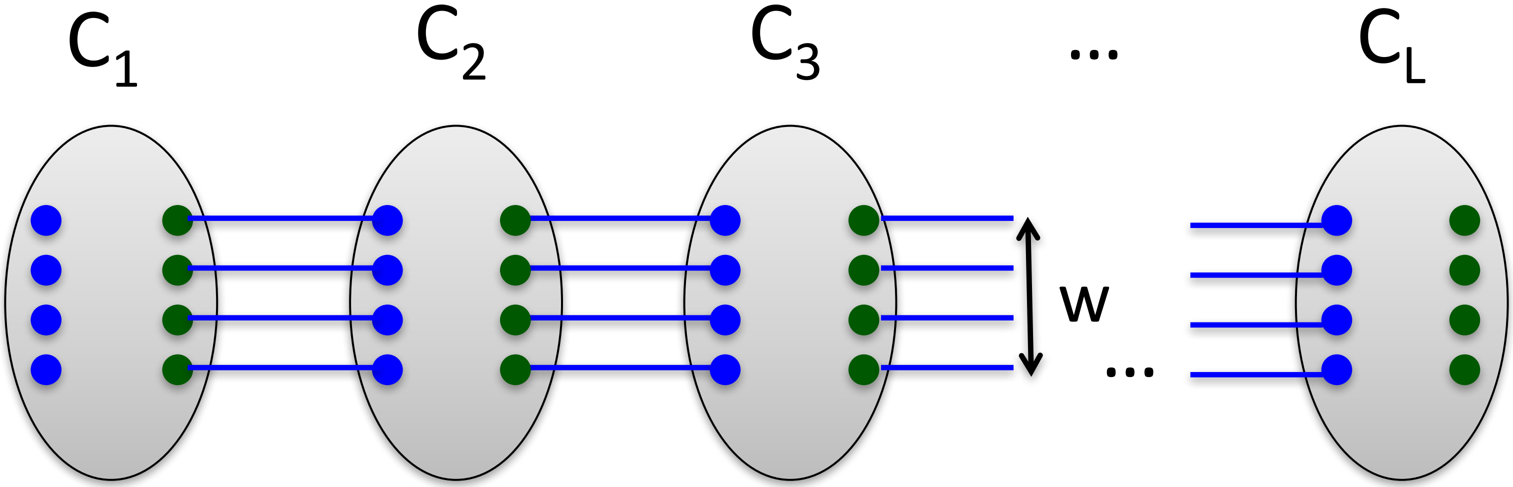
- [Robertson, Seymour, Thomas '89]: $f(k)=\Omega\left(\log^{1/5}k\right)$
- [Diestel, Gorbunov, Jensen, Thomassen '99] – simpler proof
- [Kawarabayashi, Kobayashi '12], [Leaf, Seymour '12]:

$$f(k)=\Omega\left(\sqrt{\frac{\log k}{\log\log k}}\right)$$

- [Chekuri, C '13]: $f(k)=\tilde{\Omega}\left(k^{1/98}\right)$
- [C, '16]: $f(k)=\tilde{\Omega}\left(k^{1/19}\right)$

Main Idea

width w
length L

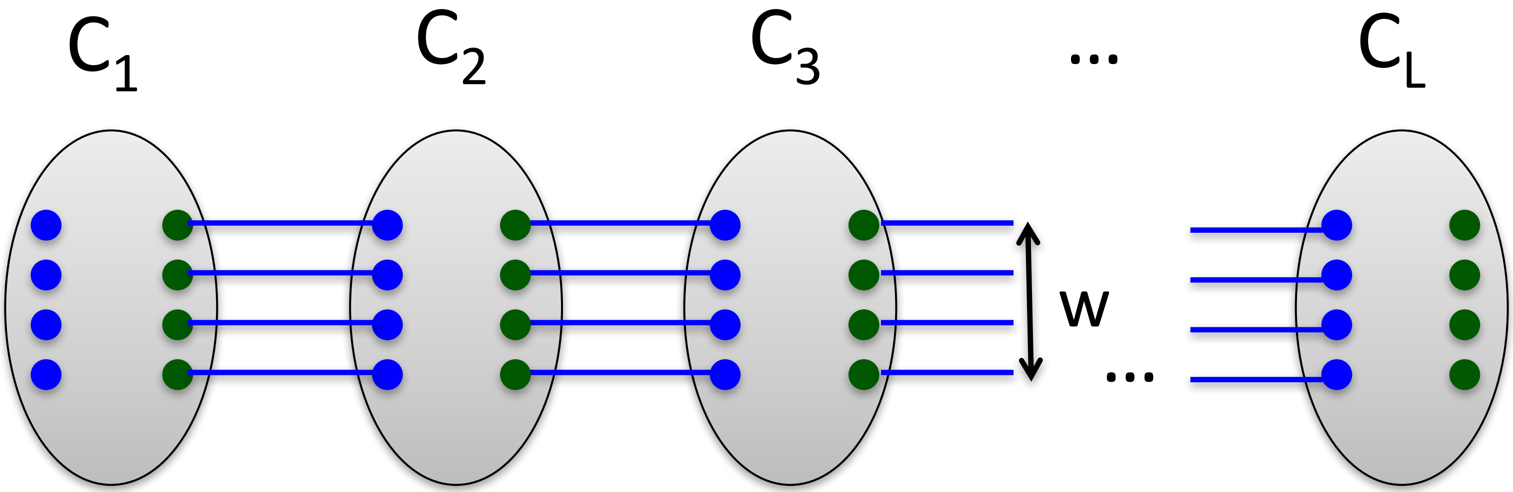


Thm: If G contains a path-of-sets system of width and length $\Theta(g^2)$, then there is a (gxg)-grid minor in G.

[Leaf, Seymour '12]
[Chekuri, C '13]

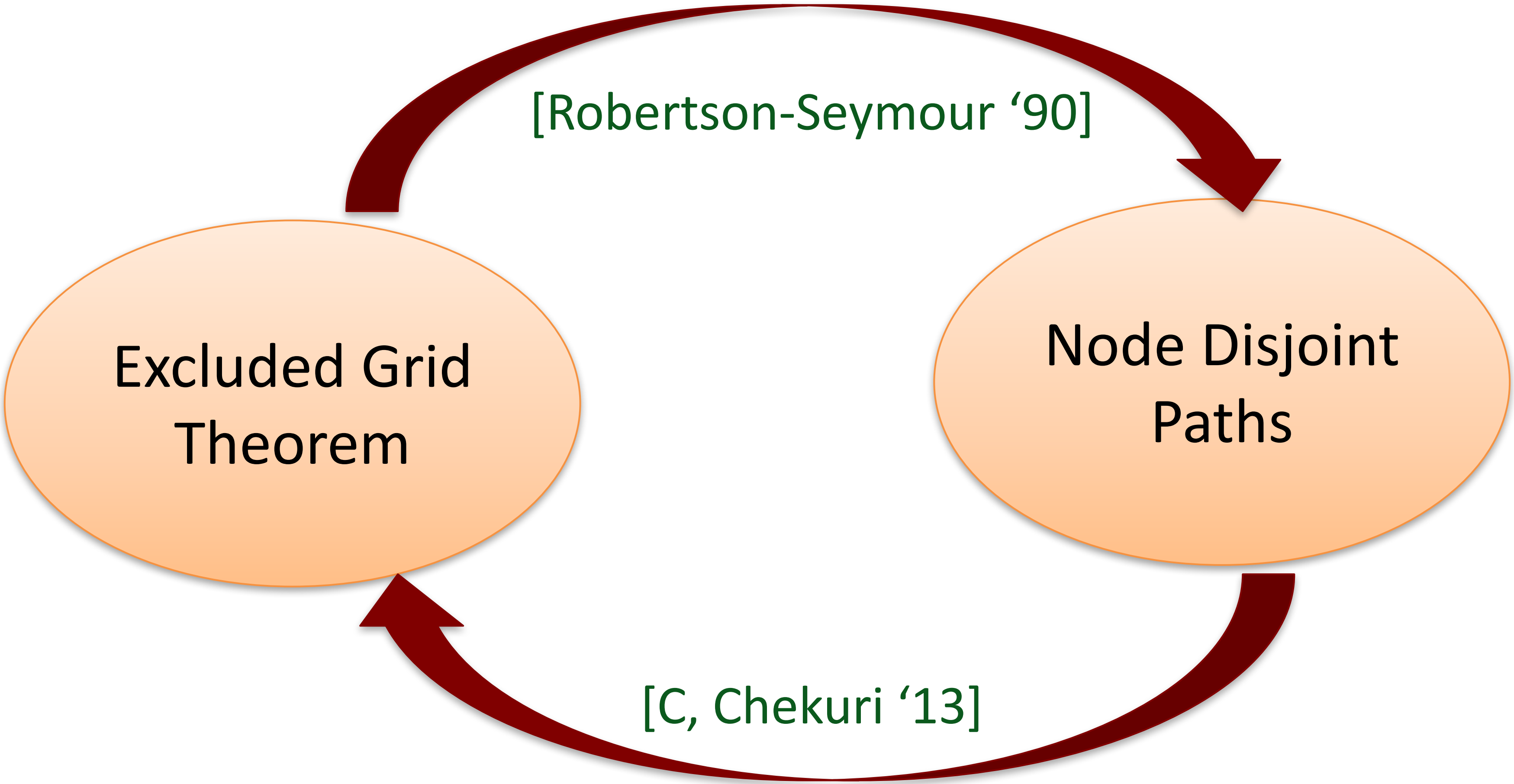
Main Idea

width w
length L



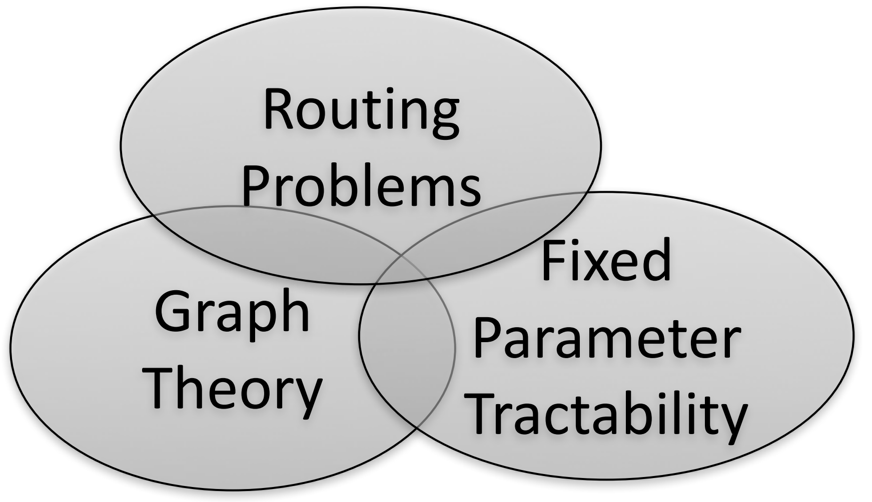
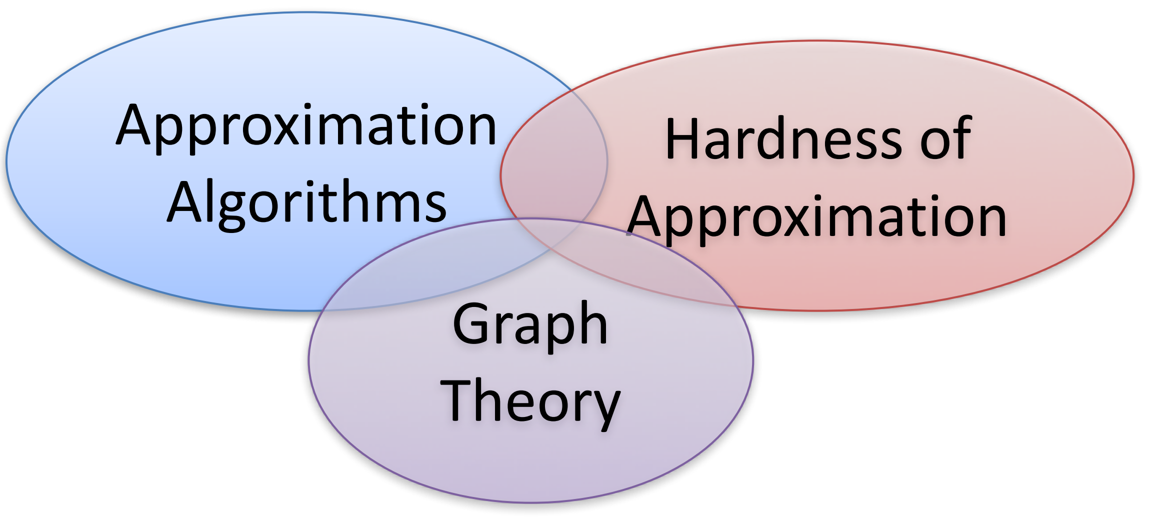
Thm: If G contains a path-of-sets system of width and length $\Theta(g^2)$, then there is a (gxg)-grid minor in G.

G has large treewidth → large well-linked set of vertices → large Path-of-Sets system


[Robertson-Seymour ‘90]
Excluded Grid Theorem
Node Disjoint Paths
[C, Chekuri ‘13]

Historical Note

- Work on routing gave slightly weaker structure than Path-of-Sets System, called Tree-of-Sets System
- We later modified it to get the Path-of-Sets system for the Excluded Grid theorem.
- This in turn helped improve results for routing problems.


Routing Problems
Graph Theory
Fixed Parameter Tractability

Approximation Algorithms
Hardness of Approximation
Graph Theory

Open Problems

- Getting tight bounds for the Excluded Grid Theorem.
- Simpler algorithms for NDP with constant k
- Congestion minimization:
  - $O(\log n / \log\log n)$-approximation algorithm
  - $\Omega(\log\log n)$-hardness of approximation
  - Integrality gap of the multicommodity LP relaxation open

Open Problems

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Planar graphs?

Thank you!