HODGE THEORY ON METRIC SPACES

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Remark. This draft modifies the previous draft (February 20, 2009) to account for a mistake pointed out to us by Thomas Schick. Our argument that the image of δ in Theorem 3 was closed was incorrect.

1 Introduction

Hodge Theory [13] studies the relationships of topology, functional analysis and geometry of a manifold. It extends the theory of the Laplacian on domains of Euclidean space or on a manifold.

However, there are a number of spaces, not manifolds, which could benefit from an extension of Hodge, and that is the motivation here. In particular we believe that a deeper analysis in the theory of vision could be led by developments of Hodge type. Spaces of images are important for developing a mathematics of vision (see e.g. Smale, Rosasco, Bouvrie, Caponnetto, and Poggio [20]; but these spaces are far from possessing manifold structures. Other settings include spaces occurring in quantum field theory, manifolds with singularities and/or non-uniform measures.

A number of previous papers have given us inspiration and guidance. For example there are those in combinatorial Hodge theory of Eckman [8], Dodziuk [7], Friedman [11], and more recently as Jiang, Lin Yao and Ye [16]. Recent decades have seen extensions of the Laplacian from its classical setting to that of combinatorial graph theory. See e.g. Fan Chung [5]. Robin Forman [10] has useful extensions from manifolds. Further extensions and relationships to the classical settings are Belkin, Niyogi [2], Belkin, De Vito, and Rosasco et al [1], Coifman, Maggioni [6], and Smale, Zhou [19].

Our approach starts with a metric space (complete, separable) X, endowed with a probability measure. For $\ell \geq 0$, an ℓ-form is a function on $\ell + 1$ tuples of points in X. The coboundary operator δ is defined from ℓ-forms to $\ell + 1$-forms in the classical way following Cech, Alexander, and Spanier. Using the L^2 adjoint δ^* of δ for a boundary operator, the ℓth order Hodge operator on ℓ-forms is defined by $\Delta_\ell = \delta^*\delta + \delta\delta^*$. The ℓ-harmonic forms on X are solutions of the equation $\Delta_\ell(f) = 0$. The ℓ-harmonic forms reflect the ℓth homology of X but have geometric features. The harmonic form is a special representative of the homology class and it may be interpreted as one satisfying an optimality condition. Moreover, the Hodge equation is linear and by choosing a finite sample from X one can obtain an approximation of this representative by a linear equation in finite dimensions.

There are two avenues to develop this Hodge theory. The first is a kernel version corresponding to a Gaussian or a reproducing kernel Hilbert space. Here the topology is trivial but the analysis gives a substantial picture. The second version is akin to the adjacency matrix of graph theory and corresponds to a threshold at a given scale alpha.

For a passage to continuous Hodge theory, one encounters:

POISSON REGULARITY PROBLEM: If $\Delta_\ell(f) = g$ is continuous under what conditions is f continuous?

It is proved that a positive solution of the Poisson Regularity Problem implies a complete Hodge decomposition for continuous ℓ-forms in the "adjacency matrix" setting (any scale alpha) provided the L^2 cohomology is finite dimensional. The problem is solved affirmatively for some cases as $\ell = 0$, or X is finite. One special case is

PROBLEM: Under what conditions are harmonic ℓ forms continuous?

Here we have a solution for $\ell = 0$ and $\ell = 1$.

When X is finite this picture overlaps with that of the combinatorial Hodge theory referred to above. The solution of these regularity problems would be progress toward the important cohomology identification problem: To what extent does the L^2 cohomology coincide with the classical cohomology?

Certain previous studies show how topology questions can give insight into the study of images. Lee, Pedersen, and Mumford [15] have investigated 3×3 pixel images from real world data bases to find the evidence for the occurrence of 1 dimensional homology classes. Moreover, Carlsson, Ishkhanov, de Silva, and Zomorodian [3] have found evidence homology of surfaces in the same data base. Here we are making an attempt to give some foundations to these studies. Moreover, this general Hodge theory could yield optimal representatives of the homology classes and provide systematic algorithms.

Some conversations with Shmuel Weinberger were helpful.

2 An L^2 Hodge Theory

In this section we construct a general Hodge Theory for certain L^2 spaces. The amount of structure needed for this theory is minimal. First, let us introduce some notation used throughout the section. X will denote a set endowed with a probability measure μ ($\mu(X) = 1$). The ℓ-fold cartesian product of X will be denoted as X^ℓ and μ_ℓ will denote the product measure on X^ℓ. Furthermore, we will assume the existence of a kernel function $K : X^2 \to \mathbf{R}$, a non-negative, measurable, symmetric function which we will assume is in $L^\infty(X \times X)$, and for certain results, we will impose additional assumptions on K. A useful example to keep in mind is this. X is a compact domain in Euclidean space, μ a Borel, but not necessarily the Euclidean measure, and K a Gaussian kernel $K(x, y) = e^{-\frac{\|x-y\|^2}{\sigma}}$, $\sigma > 0$. A simpler example is $K \equiv 1$, but the Gaussian example contains the notion of locality ($K(x, y)$ is close to 1 just when x is near y).

Recall that a chain complex of vector spaces is a sequence of vector spaces V_j and linear maps $d_j : V_j \to V_{j-1}$ such that the composition $d_{j-1} \circ d_j = 0$. A co-chain complex is the same, except that $d_j : V_j \to V_{j+1}$. The basic spaces in this section are $L^2(X^\ell)$, from which we will construct chain and co-chain complexes:

$$\cdots \xrightarrow{\partial_{\ell+1}} L^2(X^{\ell+1}) \xrightarrow{\partial_\ell} L^2(X^\ell) \xrightarrow{\partial_{\ell-1}} \cdots L^2(X) \xrightarrow{\partial_0} 0 \tag{2.1}$$

and

$$0 \to L^2(X) \xrightarrow{\delta_0} L^2(X^2) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} L^2(X^{\ell+1}) \xrightarrow{\delta_\ell} \cdots \tag{2.2}$$

Here, both ∂_ℓ and δ_ℓ will be bounded linear maps, satisfying $\partial_{\ell-1} \circ \partial_\ell = 0$ and $\delta_\ell \circ \delta_{\ell-1} = 0$. When there is no confusion, we will omit the subscripts of these operators.

We first define $\delta = \delta_{\ell-1} : L^2(X^\ell) \to L^2(X^{\ell+1})$ by

$$\delta f(x_0, \ldots, x_\ell) = \sum_{i=0}^{\ell} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell) \tag{2.3}$$

where $\hat{x}_i$ means that x_i is deleted. This is similar to the co-boundary operator of Alexander-Spanier Cohomology (see Spanier [21]). The square root in the formula is unimportant for most of the sequel, and is there so that when we define the Laplacian on $L^2(X)$, we recover the operator as defined in Gilboa and Osher [12] for example. We also note that in the case X is a finite set, δ_0 is essentially the same as the gradient operator developed by Zhou and Schölkopf [24] in the context of learning theory.

Proposition 1. *For all $\ell \geq 0$, $\delta : L^2(X^\ell) \to L^2(X^{\ell+1})$ is a bounded linear map.*

Proof. Clearly δf is measurable, as K is measurable. Since $\|K\|_\infty < \infty$, it follows from the Schwartz inequality in $\mathbf{R}^\ell$ that

$$\begin{aligned} |\delta f(x_0, \ldots, x_\ell)|^2 &\leq C \left(\sum_{i=0}^{\ell} |f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell)| \right)^2 \\ &\leq C(\ell+1) \sum_{i=0}^{\ell} |f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell)|^2 \end{aligned}$$

where $C = \|K\|_\infty^\ell$. Now, integrating both sides of the inequality with respect to $d\mu_{\ell+1}$, using Fubini's Theorem on the right side and the fact that $\mu(X) = 1$ gives us

$$\|\delta f\|_{L^2(X^{\ell+1})} \leq \sqrt{C}(\ell+1)\|f\|_{L^2(X^\ell)}$$

completing the proof.

Proposition 2. *For all $\ell \geq 1$, $\delta_\ell \circ \delta_{\ell-1} = 0$.*

Proof. For $f \in L^2(X^\ell)$ we have

$$\begin{aligned} &\delta_\ell(\delta_{\ell-1} f)(x_0, \ldots, x_{\ell+1}) \\ &= \sum_{i=0}^{\ell+1} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} (\delta_{\ell-1} f)(x_0, \ldots, \hat{x}_i, \ldots, x_{\ell+1}) \\ &= \sum_{i=0}^{\ell+1} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} \sum_{k=0}^{i-1} (-1)^k \prod_{n \neq k, i} \sqrt{K(x_k, x_n)} f(x_0, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{\ell+1}) \\ &+ \sum_{i=0}^{\ell+1} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} \sum_{k=i+1}^{\ell+1} (-1)^{k-1} \prod_{n \neq k, i} \sqrt{K(x_k, x_n)} f(x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_{\ell+1}) \end{aligned}$$

Now we note that on the right side of the second equality for given i, k, $k < i$, the corresponding term in the first sum

$$(-1)^{i+k} \prod_{j \neq i} \sqrt{K(x_i, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} f(x_0 \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{\ell+1})$$

cancels the term in the second sum where i and k are reversed

$$(-1)^{k+i-1} \prod_{j \neq k} \sqrt{K(x_k, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} f(x_0 \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{\ell+1})$$

because, as it is easily checked, using the symmetry of K that

$$\prod_{j \neq i} \sqrt{K(x_i, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)} = \prod_{j \neq k} \sqrt{K(x_k, x_j)} \prod_{n \neq k,i} \sqrt{K(x_k, x_n)}$$

It follows that (2.2) and (2.3) define a co-chain complex. We now define, for $\ell > 0$, $\partial_\ell : L^2(X^{\ell+1}) \to L^2(X^\ell)$ by

$$(2.4) \quad \partial_\ell g(x) = \sum_{i=0}^{\ell} (-1)^i \int_X \left(\prod_{j=0}^{\ell-1} \sqrt{K(t, x_j)} \right) g(x_0, \ldots, x_{i-1}, t, x_i, \ldots, x_{\ell-1})\, d\mu(t)$$

where $x = (x_0, \ldots, x_{\ell-1})$ and for $\ell = 0$ we define $\partial_0 : L^2(X) \to 0$.

Proposition 3. *For all $\ell \geq 0$, $\partial_\ell : L^2(X^{\ell+1}) \to L^2(X^\ell)$ is a bounded linear map.*

Proof. For $g \in L^2(X^{\ell+1})$, we have

$$\begin{aligned} |\partial_\ell g(x_0, \ldots, x_{\ell-1})| &\leq \|K\|_\infty^{\ell-1} \sum_{i=0}^{\ell} \int_X |g(x_0, \ldots, x_{i-1}, t, \ldots, x_{\ell-1})|\, d\mu(t) \\ &\leq \|K\|_\infty^{\ell-1} \sum_{i=0}^{\ell} \left(\int_X |g(x_0, \ldots, x_{i-1}, t, \ldots, x_{\ell-1})|^2\, d\mu(t) \right)^{\frac{1}{2}} \\ &\leq \|K\|_\infty^{\ell-1} \sqrt{\ell+1} \left(\sum_{i=0}^{\ell} \int_X |g(x_0, \ldots, x_{i-1}, t, \ldots, x_{\ell-1})|^2\, d\mu(t) \right)^{\frac{1}{2}} \end{aligned}$$

where we have used the Schwartz inequalities for $L^2(X)$ and $\mathbf{R}^{\ell+1}$ in the second and third inequalities respectively. Now, square both sides of the inequality, and integrate over X^ℓ with respect to μ_ℓ and use Fubini's Theorem arriving at the following bound to finish the proof

$$\|\partial_\ell g\|_{L^2(X^\ell)} \leq \|K\|_\infty^{\ell-1} (\ell+1) \|g\|_{L^2(X^{\ell+1})}$$

We now show that ∂_ℓ is actually the adjoint of $\delta_{\ell-1}$ (which gives a second proof of Proposition 3).

Proposition 4. *$\delta_{\ell-1}^* = \partial_\ell$. That is $< \delta_{\ell-1} f, g >_{L^2(X^{\ell+1})} = < f, \partial_\ell g >_{L^2(X^\ell)}$ for all $f \in L^2(X^\ell)$ and $g \in L^2(X^{\ell+1})$.*

Proof. For $f \in L^2(X^\ell)$ and $g \in L^2(X^{\ell+1})$ we have, by Fubini's Theorem

$$< \delta_{\ell-1} f, g > = \sum_{i=0}^{\ell} (-1)^i \int_{X^{\ell+1}} \prod_{j \neq i} \sqrt{K(x_i, x_j)} f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell) g(x_0, \ldots, x_\ell)\, d\mu_{\ell+1}$$

$$= \sum_{i=0}^{\ell} (-1)^i \int_{X^\ell} f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell) \int_X \prod_{j \neq i} \sqrt{K(x_i, x_j)} g(x_0, \ldots, x_\ell)\, d\mu(x_i)\, d\mu(x_0) \cdots \widehat{d\mu(x_i)} \cdots d\mu(x_\ell)$$

In the i-th term on the right, relabeling the variables $x_0, \ldots, \hat{x}_i, \ldots x_\ell$ with $y = (y_0, \ldots, y_{\ell-1})$ (that is $y_j = x_{j+1}$ for $j \geq i$) and putting the sum inside the integral gives us

$$\int_{X^\ell} f(y) \sum_{i=0}^{\ell} (-1)^i \int_X \prod_{j=0}^{\ell-1} \sqrt{K(x_i, y_j)} g(y_0, \ldots, y_{i-1}, x_i, y_i, \ldots, y_{\ell-1})\, d\mu(x_i)\, d\mu_\ell(y)$$

which is just $< f, \partial_\ell g >$.

We note, as a corollary, that $\partial_{\ell-1} \circ \partial_\ell = 0$, and thus (2.1) and (2.4) define a chain complex. We can thus define the homology and cohomology spaces (real coefficients) of (2.1) and (2.2) as follows. Since $\mathrm{Im}\partial_\ell \subset \mathrm{Ker}\partial_{\ell-1}$ and $\mathrm{Im}\delta_{\ell-1} \subset \mathrm{Ker}\delta_\ell$ we define the quotient spaces

$$H_\ell(X) = H_\ell(X, K, \mu) = \frac{\mathrm{Ker}\partial_\ell}{\mathrm{Im}\partial_{\ell-1}} \qquad H^\ell(X) = H^\ell(X, K, \mu) = \frac{\mathrm{Ker}\delta_\ell}{\mathrm{Im}\delta_{\ell-1}} \tag{2.5}$$

which will be referred to the L^2 ℓ-dimensional homology and cohomology respectively. In later sections, with additional assumptions on X and K, we will investigate the relation between these spaces and the topology of X for example, the Alexander-Spanier cohomology. In order to proceed with the Hodge Theory, we consider δ to be the analogue of the exterior derivative d on ℓ forms from differential topology, and $\partial = \delta^*$ as the analogue of d^*. We then define the Laplacian (in analogy with the Hodge Laplacian) to be $\Delta_\ell = \delta_\ell^* \delta_\ell + \delta_{\ell-1} \delta_{\ell-1}^*$. Clearly $\Delta_\ell : L^2(X^{\ell+1}) \to L^2(X^{\ell+1})$ is a bounded, self adjoint, positive semi-definite operator since for $f \in L^2(X^{\ell+1})$

$$< \Delta f, f > = < \delta^* \delta f, f > + < \delta \delta^* f, f > = \|\delta f\|^2 + \|\delta^* f\|^2 \tag{2.6}$$

where we have left of the subscripts on the operators. The Hodge Theorem will give a decomposition of $L^2(X^{\ell+1})$ in terms of the image spaces under δ, δ^* and the kernel of Δ, and also identify the kernel of Δ with $H^\ell(X, K, \mu)$. Elements of the kernel of Δ will be referred to as harmonic. For $\ell = 0$, one easily computes that

$$\frac{1}{2} \Delta_0 f(x) = D(x) f(x) - \int_X K(x, y) f(y)\, d\mu(y) \quad \text{where} \quad D(x) = \int_X K(x, y)\, d\mu(y)$$

which, in the case K is a positive definite kernel on X is the Laplacian defined in Smale and Zhou [19] (see section 5)

Remark. It follows from (2.6) that $\Delta f = 0$ if and only if $\delta_\ell f = 0$ and $\delta_\ell^* f = 0$, and so $\mathrm{Ker}\Delta_\ell \subset \mathrm{Ker}\delta_\ell$.

The main goal of this section is the following L^2 Hodge theorem:

Theorem 1. *Assume that $0 < \sigma \leq K(x,y) \leq \|K\|_\infty < \infty$ almost everywhere. Then we have the orthogonal, direct sum decomposition*

$$L^2(X^{\ell+1}) = Im\delta_{\ell-1} \oplus Im\delta_\ell^* \oplus Ker\Delta_\ell$$

and the cohomology space $H^\ell(X,K,\mu)$ is isomorphic to $Ker\Delta_\ell$, with each equivalence class in the former having a unique representative in the latter.

In this case $H^\ell(X) = 0$ for $\ell > 0$ and $H^0(x) = \mathbf{R}$. Indeed, the theorem holds as long as δ_ℓ (or equivalently) ∂_ℓ) has closed range for all ℓ.

In subsequent sections we will have occasion to use the L^2 spaces of alternating functions:

$$\begin{aligned} L_a^2(X^{\ell+1}) =&\{f \in L^2(X^{\ell+1}) : f(x_0,\ldots,x_\ell) = (-1)^{\mathrm{sign}\sigma} f(x_{\sigma(x_0)},\ldots,x_{\sigma(x_\ell)}),\\ &\sigma \text{ a permutation}\}\end{aligned}$$

Due to the symmetry of K, it is easy to check that the coboundary δ preserves the alternating property, and thus propositions 1 through 4, as well as formulas (2.1), (2.2), (2.5) and (2.6) hold with L_a^2 in place of L^2. We note that the alternating map

$$\mathrm{Alt} : L^2(X^{\ell+1}) \to L_a^2(X^{\ell+1})$$

defined by

$$\mathrm{Alt}(f)(x_0,\ldots,x_\ell) = \frac{1}{(\ell+1)!} \sum_{\sigma \in S_{\ell+1}} (-1)^{\mathrm{sign}\sigma} f(x_{\sigma(x_0)},\ldots,x_{\sigma(x_\ell})$$

is a projection relating the two definitions of ℓ-forms.

We first collect some relevant facts in a more abstract setting in the following

Lemma 1 (Hodge Lemma). *Suppose we have the cochain and corresponding dual chain complexes*

$$0 \to V_0 \xrightarrow{\delta_0} V_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} V_\ell \xrightarrow{\delta_\ell} \cdots$$

$$\cdots \xrightarrow{\delta_\ell^*} V_\ell \xrightarrow{\delta_{\ell-1}^*} V_{\ell-1} \xrightarrow{\delta_{\ell-2}^*} \cdots \xrightarrow{\delta_0^*} V_0 \to 0$$

where for $\ell = 0,1,\ldots,$ $V_\ell, <,>_\ell$ is a Hilbert space, δ_ℓ (and thus δ_ℓ^*, the adjoint of δ_ℓ) is a bounded linear map with $\delta^2 = 0$. Let $\Delta_\ell = \delta_\ell^*\delta_\ell + \delta_{\ell-1}\delta_{\ell-1}^*$. Then the following are equivalent

1. *δ_ℓ has closed range for all ℓ*
2. *δ_ℓ^* has closed range for all ℓ*

Furthermore, if one of the above conditions hold, we have the orthogonal, direct sum decomposition into closed subspaces

$$V_\ell = Im\delta_{\ell-1} \oplus Im\delta_\ell^* \oplus Ker\Delta_\ell$$

and the quotient space $\frac{\mathrm{Ker}\delta_\ell}{\mathrm{Im}\delta_{\ell-1}}$ *is isomorphic to Ker*Δ_ℓ*, with each equivalence class in the former having a unique representative in the latter.*

Proof. We first assume conditions 1 and 2 above and prove the decomposition. For all $f \in V_{\ell-1}$ and $g \in V_{\ell+1}$ we have

$$< \delta_{\ell-1} f, \delta_\ell^* g >_\ell = < \delta_\ell \delta_{\ell-1} f, g >_{\ell+1} = 0.$$

Also, as in (2.6), $\Delta_\ell f = 0$ if and only if $\delta_\ell f = 0$ and $\delta_{\ell-1}^* f = 0$. Therefore, if $f \in \mathrm{Ker}\Delta_\ell$, then for all $g \in V_{\ell-1}$ and $h \in V_{\ell+1}$

$$< f, \delta_{\ell-1} g >_\ell = < \delta_{\ell-1}^* f, g >_{\ell-1} = 0 \quad \text{and} \quad < f, \delta_\ell^* h >_\ell = < \delta_\ell f, h >_{\ell+1} = 0$$

and thus $\mathrm{Im}\delta_{\ell-1}$, $\mathrm{Im}\delta_\ell^*$ and $\mathrm{Ker}\Delta_\ell$ are mutually orthogonal. Now, since $\mathrm{Im}\delta_{\ell-1} \oplus \mathrm{Im}\delta_\ell^*$ is closed, to prove the decomposition it suffices to show that $\mathrm{Ker}\Delta_\ell \supseteq (\mathrm{Im}\delta_{\ell-1} \oplus \mathrm{Im}\delta_\ell^*)^\perp$. Let $v \in (\mathrm{Im}\delta_{\ell-1} \oplus \mathrm{Im}\delta_\ell^*)^\perp$. Then, for all $w \in V_\ell$, $< \delta_\ell v, w > = < v, \delta_\ell^* w > = 0$ and $< \delta_{\ell-1}^* v, w > = < v, \delta_{\ell-1} w > = 0$, which implies that $\delta_\ell v = 0$ and $\delta_{\ell-1}^* v = 0$ and as noted above this implies that $\Delta_\ell v = 0$, proving the decomposition.

We define an isomorphism

$$\tilde{P} : \frac{\mathrm{Ker}\delta_\ell}{\mathrm{Im}\delta_{\ell-1}} \to \mathrm{Ker}\Delta_\ell$$

as follows. Let $P : V_\ell \to \mathrm{Ker}\Delta_\ell$ be the orthogonal projection. Then, for an equivalence class $[f] \in \frac{\mathrm{Ker}\delta_\ell}{\mathrm{Im}\delta_{\ell-1}}$ define $\tilde{P}([f]) = P(f)$. Note that if $[f] = [g]$ then $f = g + h$ with $h \in \mathrm{Im}\delta_{\ell-1}$, and therefore $P(f) - P(g) = P(h) = 0$ by the orthogonal decomposition, and so $\tilde{P}$ is well defined, and linear as P is linear. If $\tilde{P}([f]) = 0$ then $P(f) = 0$ and so $f \in \mathrm{Im}\delta_{\ell-1} \oplus \mathrm{Im}\delta_\ell^*$. But $f \in \mathrm{Ker}\delta_\ell$, and so, for all $g \in V_{\ell+1}$ we have $< \delta_\ell^* g, f > = < g, \delta_\ell > = 0$, and thus $f \in \mathrm{Im}\delta_{\ell-1}$ and therefore $[f] = 0$ and $\tilde{P}$ is injective. On the other hand, $\tilde{P}$ is surjective because, if $w \in \mathrm{Ker}\Delta_\ell$, then $w \in \mathrm{Ker}\delta_\ell$ and so $\tilde{P}([w]) = P(w) = w$.

Finally, the equivalence of conditions 1 and 2, is a general fact about Hilbert spaces. If $\delta : V \to H$ is a bounded linear map between Hilbert spaces, and δ^* is it's adjoint, and if $\mathrm{Im}\delta$ is closed in H, then $\mathrm{Im}\delta^*$ is closed in V. We include the proof for completeness. Since $\mathrm{Im}\delta$ is closed, the bijective map

$$\delta : (\mathrm{Ker}\delta)^\perp \to \mathrm{Im}\delta$$

is an isomorphism by the open mapping theorem. It follows that

$$\inf\{\|\delta(v)\| : v \in (\mathrm{Ker}\delta)^\perp,\ \|v\| = 1\} > 0$$

Since $\mathrm{Im}\delta \subset (\mathrm{Ker}\delta^*)^\perp$, it suffices to show that

$$\delta^*\delta : (\mathrm{Ker}\delta)^\perp \to (\mathrm{Ker}\delta)^\perp$$

is an isomorphism, for then $\mathrm{Im}\delta^* = (\mathrm{Ker}\delta)^\perp$ which is closed. However, this is established by noting that $< \delta^*\delta v, v > = \|\delta v\|^2$ and the above inequality imply that

$$\inf\{< \delta^*\delta v, v > : v \in (\mathrm{Ker}\delta)^\perp,\ \|v\| = 1\} > 0.$$

This finishes the proof of the lemma.

Corollary. *For all $\ell \geq 0$ the following are isomorphisms*

$$\delta_\ell : Im\delta_\ell^* \to Im\delta_\ell \qquad and \qquad \delta_\ell^* : Im\delta_\ell \to Im\delta_\ell^*$$

Proof. The first map in injective because if $\delta(\partial f) = 0$ then $0 =< \delta\partial f, f >= \|f\|^2$ and so $\delta f = 0$. It is surjective because of the decomposition (leaving out the subscripts)

$$\delta(V) = \delta(\mathrm{Im}\delta \oplus \mathrm{Im}\delta^* \oplus \mathrm{Ker}\Delta) = \delta(\mathrm{Im}\delta^*)$$

since δ is zero on the first and third summands of the left side of the second equality. The argument for the second map is the same.

The difficulty in applying the Hodge Lemma is in verifying that either δ or δ^* has closed range. A sufficient condition is the following, first pointed out to us by Shmuel Weinberger.

Proposition. *Suppose that in the context of Lemma 1, the L^2 cohomology space $Ker\delta_\ell/Im\delta_{\ell-1}$ is finite dimensional. Then $\delta_{\ell-1}$ has closed range.*

Proof. We show more generally, that if $T : B \to V$ is a bounded linear map with $\mathrm{Im}T$ having finite codimension in V then $\mathrm{Im}T$ is closed in V. We can assume without loss of generality that T is injective, by replacing B with $(\mathrm{Ker}T)^\perp$ if necessary. Thus $T : B \to \mathrm{Im}T \oplus F = V$ where $\dim F < \infty$. Now define $G : B \oplus F \to V$ by $G(x, y) = Tx+y$. G is bounded , surjective and injective, and thus an isomorphism by the open mapping theorem. Therefore $G(B) = T(B)$ is closed in V.

We now finish the proof of Theorem 1. Consider first the special case where $K(x, y) = 1$ for all x, y in X. Let ∂_ℓ^0 be the corresponding operator in (2.4). We have

Lemma 2. *For $\ell > 1$, $Im\partial_\ell^0 = Ker\partial_{\ell-1}^0$, and $Im\partial_1^0 = \{1\}^\perp$ the orthogonal complement of the constants in $L^2(X)$.*

Of course this implies that $\mathrm{Im}\partial_\ell$ is closed for all ℓ since null spaces and orthogonal complements are closed, and in fact shows that the homology (2.5) in this case is trivial for $\ell > 0$ and one dimensional for $\ell = 0$.

Proof of Lemma 2. Let $h \in \{1\}^\perp \subset L^2(X)$. Define $g \in L^2(X^2)$ by $g(x, y) = h(y)$. Then from (2.4)

$$\partial_1^0 g(x_0) = \int_X (g(t, x_0) - g(x_0, t))\, d\mu(t) = \int_X (h(x_0) - h(t))\, d\mu(t) = h(x_0)$$

since $\mu(X) =$ and $\int_X h\, d\mu = 0$. It can be easily checked that ∂_1^0 maps $L^2(X^2)$ into $\{1\}^\perp$, thus proving the lemma for $\ell = 1$. For $\ell > 1$ let $h \in \mathrm{Ker}\partial_{\ell-1}^0$. Define $g \in L^2(X^{\ell+1})$ by $g(x_0, \ldots, x_\ell) = (-1)^\ell h(x_0, \ldots, x_{\ell-1})$. Then, by (2.4)

$$\begin{aligned}
\partial_\ell^0 g(x_0, \ldots, x_{\ell-1}) &= \sum_{i=0}^{\ell} (-1)^i \int_X g(x_0, \ldots, x_{i-1}, t, x_i, \ldots, x_{\ell-1})\, d\mu(t) \\
&= (-1)^\ell \sum_{i=0}^{\ell-1} (-1)^i \int_X h(x_0, \ldots, x_{i-1}, t, x_i, \ldots, x_{\ell-2})\, d\mu(t) \\
&+ (-1)^{2\ell} h(x_0, \ldots, x_{\ell-1}) \\
&= (-1)^\ell \partial_{\ell-1}^0 h(x_0, \ldots, x_{\ell-2}) + h(x_0, \ldots, x_{\ell-1}) \\
&= h(x_0, \ldots, x_{\ell-1})
\end{aligned}$$

since $\partial^0_{\ell-1} h = 0$, finishing the proof.

The next lemma give some general conditions on K that guarantee ∂_ℓ has closed range.

Lemma 3. *Assume that $K(x,y) \geq \sigma > 0$ for all $x, y \in X$. Then $Im\partial_\ell$ is closed for all ℓ. In fact, $Im\partial_\ell = Ker\partial_{\ell-1}$ for $\ell > 1$ and has co-dimension one in $L^2(X)$ for $\ell = 1$.*

Proof. Let $M_\ell : L^2(X^\ell) \to L^2(X^\ell)$ be the multiplication operator

$$M_\ell(f)(x_0, \dots, x_\ell) = \prod_{j \neq k} \sqrt{K(x_j, x_k)} f(x_0, \dots, x_\ell)$$

. Since $K \in L^\infty(X^2)$ and is bounded below by σ, M_ℓ clearly defines an isomorphism. The Lemma then follows from Lemma 2, and the observation that

$$\partial_\ell = M^{-1}_{\ell-1} \circ \partial^0_\ell \circ M_\ell$$

Theorem1 now follows from the Hodge Lemma, and Lemma 3.

We also note that Lemma 2, Lemma 3 and Theorem 1 hold in case the alternating setting, when $L^2(X^\ell)$ is replaced with $L^2_a(X^\ell)$.

For background, one could see Munkres [16] for the algebraic topology, Lang [14] for the analysis, and Warner [22] for the geometry.

3 Metric Spaces

For the rest of the paper, we assume that X is a complete, separable metric space, and μ is a Borel probability measure on X, and K is a continuous function on X^2 (as well as symetric, nonnegative and bounded as in section 2). We will also assume throughout the rest of the paper that $\mu(U) > 0$ for U any nonempty open set.

The goal of this section is a Hodge Decomposition for continuous alternating functions. Let $C(X^{\ell+1})$ denote the continuous functions on $X^{\ell+1}$. We will use the following notation:

$$C^{\ell+1} = C(X^{\ell+1}) \cap L^2_a(X^{\ell+1}) \cap L^\infty(X^{\ell+1})$$

Note that

$$\delta : C^{\ell+1} \to C^{\ell+2} \quad \text{and} \quad \partial : C^{\ell+1} \to C^\ell$$

are well defined linear maps. The only thing to check is that $\delta(f)$ and $\partial(f)$ are continuous and bounded if $f \in C^{\ell+1}$. In the case of $\delta(f)$ this is obvious from (2.3). The following proposition from analysis, (2.4) and the fact that μ is Borel imply that $\partial(f)$ is bounded and continuous.

Proposition. *Let Y and X be metric spaces, μ a Borel measure on X, and $M, g \in C(Y \times X) \cap L^\infty(Y \times X)$. Then $dg \in C(X) \cap L^\infty(X)$, where*

$$dg(x) = \int_X M(x,t) g(x,t) \, d\mu(t)$$

Proof. The fact that dg is bounded follows easily from the definition and properties of M and g, and continuity follows from a simple application of the Dominated Convergence Theorem, proving the proposition.

Therefore we have the chain complexes:

$$\cdots \xrightarrow{\partial_{\ell+1}} C^{\ell+1} \xrightarrow{\partial_\ell} C^\ell \xrightarrow{\partial_{\ell-1}} \cdots C^1 \xrightarrow{\partial_0} 0 \tag{3.1}$$

and

$$0 \longrightarrow C^1 \xrightarrow{\delta_0} C^2 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{\ell-1}} C^{\ell+1} \xrightarrow{\delta_\ell} \cdots \tag{3.2}$$

In this setting we will prove

Theorem 2. *Assume that K satisfies the hypotheses of Theorem 1, and is continuous. Then we have the orthogonal, direct sum decomposition*

$$C^{\ell+1} = \delta(C^\ell) \oplus \partial(C^{\ell+2}) \oplus Ker_C\Delta$$

where $Ker_C\Delta$ denotes the subspace of elements in $Ker\Delta$ that are in $C^{\ell+1}$.

As in Theorem 1, the third summand is trivial except when $\ell = 0$ in which case it consists of the constant functions. We first assume that $K \equiv 1$. The proof follows from a few propositions. In the remainder of the section, $\mathrm{Im}\delta$ and $\mathrm{Im}\partial$ will refer to the image spaces of δ and ∂ as operators on L^2_a. The next proposition gives formulas for ∂ and Δ on alternating functions.

Proposition 5. *For $f \in L^2_a(X^{\ell+1})$ we have*

$$\partial f(x_0, \ldots, x_{\ell-1}) = (\ell+1) \int_X f(t, x_0, \ldots, x_{\ell-1})\, d\mu(t)$$

and

$$\Delta f(x_0, \ldots, x_\ell) = (\ell+2) f(x_0, \ldots, x_\ell) - \frac{1}{\ell+1} \sum_{i=0}^{\ell} \partial f(x_0, \ldots, \hat{x}_i, \ldots, x_\ell)$$

Proof. The first formula follows immediately from (2.4) and the fact that f is alternating. The second follows from a simple calculation using (2.3), (2.4) and the fact that f is alternating.

Let P_1, P_2, and P_3 be the orthogonal projections implicit in Theorem 1

$$P_1 : L^2_a(X^{\ell+1}) \to \mathrm{Im}\delta,\ P_2 : L^2_a(X^{\ell+1}) \to \mathrm{Im}\partial,\ \text{and}\ P_3 : L^2_a(X^{\ell+1}) \to \mathrm{Ker}\Delta$$

Proposition 6. *Let $f \in C^{\ell+1}$. Then $P_1(f) \in C^{\ell+1}$*

Proof. It suffices to show that $P_1(f)$ is continuous and bounded. Let $g = P_1(f)$. It follows from Theorem 1 that $\partial f = \partial g$, and therefore ∂g is continuous and bounded. Since $\delta g = 0$, we have, for $t, x_0, \ldots, x_\ell \in X$

$$0 = \delta g(t, x_0, \ldots, x_\ell) = g(x_0, \ldots, x_\ell) - \sum_{i=0}^{\ell} (-1)^i g(t, x_0, \ldots, \hat{x}_i, \ldots, x_\ell)$$

Integrating over $t \in X$ gives us

$$g(x_0, \dots, x_\ell) = \int_X g(x_0, \dots, x_\ell)\, d\mu(t) = \sum_{i=0}^{\ell} (-1)^i \int_X g(t, x_0, \dots, \hat{x}_i, \dots, x_\ell)\, d\mu(t)$$
$$= \frac{1}{\ell + 1} \sum_{i=0}^{\ell} (-1)^i \partial g(x_0, \dots, \hat{x}_i, \dots, x_\ell)$$

As ∂g is continuous and bounded, this implies g is continuous and bounded.

Corollary. *If $f \in C^{\ell+1}$, then $P_2(f) \in C^{\ell+1}$.*

This follows from the Hodge decomposition (Theorem 1) and the fact that $P_3(f)$ is continuous and bounded (being a constant).

The following proposition can be thought of as analogous to a regularity result in elliptic PDE's. It states that solutions to $\Delta u = f$, f continuous, which are apriori in L^2 are actually continuous.

Proposition 7. *If $f \in C^{\ell+1}$ and $\Delta u = f$, $u \in L^2_a(X^{\ell+1})$ then $u \in C^{\ell+1}$.*

Proof. From Proposition 5, (with u in place of f) we have

$$\Delta u(x_0, \dots, x_\ell) = (\ell + 2) u(x_0, \dots, x_\ell) - \frac{1}{\ell + 1} \sum_{i=0}^{\ell} \partial u(x_0, \dots, \hat{x}_i, \dots, x_\ell)$$
$$= f(x_0, \dots, x_\ell)$$

and solving for u, we get

$$u(x_0, \dots, x_\ell) = \frac{1}{\ell + 2} f(x_0, \dots, x_\ell) + \frac{1}{(\ell + 2)(\ell + 1)} \sum_{i=0}^{\ell} \partial u(x_0, \dots, \hat{x}_i, \dots, x_\ell)$$

It therefore suffices to show that ∂u is continuous and bounded. However, it is easy to check that $\Delta \circ \partial = \partial \circ \Delta$ and thus

$$\Delta(\partial u) = \partial \Delta u = \partial f$$

is continuous and bounded. But then, again using Proposition 5

$$\Delta(\partial u)(x_0, \dots, x_{\ell-1}) = (\ell + 1) \partial u(x_0, \dots, x_{\ell-1})$$
$$- \frac{1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \partial(\partial u)(x_0, \dots, \hat{x}_i, \dots, x_{\ell-1})$$

and so, using $\partial^2 = 0$ we get

$$(\ell + 1) \partial u = \partial f$$

which implies that ∂u is continuous and bounded, finishing the proof.

Proposition 8. *If $g \in C^{\ell+1} \cap Im\delta$, then $g = \delta h$ for some $h \in C^\ell$.*

Proof. From the corollary of the Hodge Lemma, let h be the unique element in $\mathrm{Im}\partial$ with $g = \delta h$. Now ∂g is continuous and bounded, and

$$\partial g = \partial \delta h = \partial \delta h + \delta \partial h = \Delta h$$

since $\partial h = 0$. But now h is continuous and bounded from Proposition 7.

Proposition 9. *If $g \in C^{\ell+1} \cap L^2_a(X^{\ell+1})$, the $g = \partial h$ for some $h \in C^{\ell+2}$.*

The proof is identical to the one for Proposition 8.

Theorem 2, in the case $K \equiv 1$ now follows from Propositions 6 through 9. The proof easily extends to general K.

4 Hodge Theory at Scale α

As seen in sections 2 and 3, the chain and cochain complexes constructed on the whole space yield trivial cohomology groups. In order to have a theory that gives us topological information about X, we define our complexes on a neighborhood of the diagonal, and restrict the boundary and coboundary operator to these complexes. The corresponding cohomology can be considered a scaled cohomology of X, with the scale being the size of the neighborhood. We will assume throughout this section that (X,d) is a compact metric space. For $x,y \in X^\ell$, $\ell > 1$, this induces a metric compatible with the product topology

$$d_\ell(x,y) = max\{d(x_0,y_0),\ldots d(x_{\ell-1},y_{\ell-1})\}$$

The diagonal D_ℓ of X^ℓ is just $\{x \in X^\ell : x_i = x_j,\ i,j = 0,\ldots,\ell-1\}$ For $\alpha > 0$ we define the α neighborhood of the diagonal to be

$$\begin{aligned} U_\alpha^\ell &= \{x \in X^\ell : d_\ell(x, D_\ell) \leq \alpha\} \\ &= \{x \in X^\ell : \exists t \in X \text{ such that } d(x_i,t) \leq \alpha,\ i = 0,\ldots,\ell-1\} \end{aligned}$$

Observe that U_α^ℓ is closed and that for $\alpha \geq$ diameter X, $U_\alpha^\ell = X^\ell$.

The measure μ_ℓ induces a Borel measure on U_α^ℓ which we will simply denote by μ_ℓ (not a probability measure). For simplicity, we will take $K \equiv 1$ throughout this section, and consider only alternating functions in our complexes. We first discuss the L^2 theory, and thus our basic spaces will be $L_a^2(U_\alpha^\ell)$, the space of alternating functions on U_α^ℓ that are in L^2 with respect to μ_ℓ, $\ell > 0$. Note that if $(x_0,\ldots,x_\ell) \in U_\alpha^{\ell+1}$, then $(x_0,\ldots,\hat{x}_i,\ldots,x_\ell) \in U_\alpha^\ell$ for $i = 0,\ldots,\ell$. It follows that if $f \in L_a^2(U_\alpha^\ell)$, then $\delta f \in L_a^2(U_\alpha^{\ell+1})$. We therefore have the well defined cochain complex

$$0 \longrightarrow L_a^2(U_\alpha^1) \xrightarrow{\delta} L_a^2(U_\alpha^2) \xrightarrow{\delta} \cdots \xrightarrow{\delta} L_a^2(U_\alpha^\ell) \xrightarrow{\delta} L_a^2(U_\alpha^{\ell+1}) \cdots$$

Since $\partial = \delta^*$ depends on the integral, the expression for it will be different than (2.4). We define a "slice" by

$$S_{x_0\cdots x_{\ell-1}} = \{t \in X : (x_0,\ldots,x_{\ell-1},t) \in U_\alpha^{\ell+1}\}$$

We note that, for $S_{x_0\cdots x_{\ell-1}}$ to be nonempty, $(x_0,\ldots,x_{\ell-1})$ must be in U_α^ℓ, and furthermore

$$U_\alpha^{\ell+1} = \{(x_0,\ldots,x_\ell) : (x_0,\ldots,x_{\ell-1}) \in U_\alpha^\ell, \text{ and } x_\ell \in S_{x_0\cdots x_{\ell-1}}\}$$

It follows from the proof of Proposition 1(section 2) and the fact that $K \equiv 1$, that $\delta : L_a^2(U_\alpha^\ell) \to L_a^2(U_\alpha^{\ell+1})$ is bounded and that $\|\delta\| \leq \ell+1$, and therefore it's adjoint is bounded. The adjoint of the operator $\delta : L_a^2(U_\alpha^\ell) \to L_a^2(U_\alpha^{\ell+1})$ will be denoted, as before, by either ∂ or δ^* (without the subscript ℓ).

Proposition 10. *For $f \in L_a^2(U_\alpha^{\ell+1})$ we have*

$$\partial f(x_0,\ldots,x_{\ell-1}) = (\ell+1)\int_{S_{x_0\cdots x_{\ell-1}}} f(t,x_0,\ldots,x_{\ell-1})\, d\mu(t)$$

Proof. The proof is essentially the same as the proof of Proposition 4, using the fact that $K \equiv 1$, f is alternating, and the above remark.

It is worth noting that the domain of integration depends on $x \in U^\ell_\alpha$, and this makes the subsequent analysis more difficult than in section 3. We thus have the corresponding chain complex

$$\cdots \xrightarrow{\partial} L^2_a(U^{\ell+1}_\alpha) \xrightarrow{\partial} L^2_a(U^\ell_\alpha) \xrightarrow{\partial} \cdots L^2_a(U^1_\alpha) \xrightarrow{\partial} 0$$

Of course, $U^1_\alpha = X$. The corresponding Hodge Laplacian in this setting is $\Delta : L^2_a(U^\ell_\alpha) \to L^2_a(U^\ell_\alpha)$ is $\Delta = \partial\delta + \delta\partial$, where all of these operators depend on ℓ and α. When we want to emphasize this dependence, we will list ℓ and (or) α as subscripts. We will use the following notation for the cohomology and harmonic functions of the above complexes:

$$H^\ell_{L^2,\alpha}(X) = \frac{\text{Ker } \delta_{\ell,\alpha}}{\text{Im } \delta_{\ell-1,\alpha}} \quad \text{and} \quad \text{Harm}^\ell_\alpha(X) = \text{Ker } \Delta_{\ell,\alpha}$$

Remark. If $\alpha \geq \text{diam } X$, then $U^\ell_\alpha = X^\ell$, so the situation is as in Theorem 1 of section 2, so $H^\ell_{L^2,\alpha}(X) = 0$ for $\ell > 0$ and $H^0_{L^2,\alpha}(X) = \mathbf{R}$. Also, if X is a finite union of connected components $X_1, \ldots, X_k$, and $\alpha < \text{dist}(X_i, X_j)$ for all $i \neq j$, then $H^\ell_{L^2,\alpha}(X) = \oplus^k_{i=1} H^\ell_{L^2,\alpha}(X_i)$.

Theorem 3. *If X is a compact metric space, $\alpha > 0$, and X satisfies the L^2 cohomology spaces $Ker\delta_{\ell,\alpha}/Im\delta_{\ell-1,\alpha}$, $\ell \geq 0$ are finite dimensional, then we have the orthogonal direct sum decomposition into closed subspaces*

$$L^2_a(U^\ell_\alpha) = \mathit{Im}\ \delta_{\ell-1} \oplus \mathit{Im}\ \delta^*_\ell \oplus \mathit{Harm}^\ell_\alpha(X) \quad \mathit{all}\ \ell$$

Furthermore, $H^\ell_{\alpha,L^2}(X)$ is isomorphic to $Harm^\ell_\alpha(X)$, with each equivalence class in the former having a unique representative in the latter.

Proof. This is immediate from the Hodge Lemma (Lemma 1). One only needs that $\text{Im}\delta_{\ell,\alpha}$ is closed, which follows from the unnumbered proposition 0f section 2.

We record the formulas for $\delta\partial f$ and $\partial\delta f$ for $f \in L^2_a(U^{\ell+1}_\alpha)$

$$\begin{aligned}(4.3)\quad & \delta(\partial f)(x_0, \ldots, x_\ell) \\ & \qquad = (\ell+1) \sum_{i=0}^{\ell} (-1)^i \int_{S_{x_0,\ldots,\hat{x}_i,\ldots,x_\ell}} f(t, x_0, \ldots, \hat{x}_i, \ldots, x_\ell) d\mu(t)\end{aligned}$$

$$\begin{aligned}(4.4)\quad \partial(\delta f)(x_0, \ldots, x_\ell) &= (\ell+2)\mu(S_{x_0,\ldots,x_\ell}) f(x_0, \ldots, x_\ell) \\ &+ (\ell+2) \sum_{i=0}^{\ell} (-1)^{i+1} \int_{S_{x_0,\ldots,x_\ell}} f(t, x_0, \ldots, \hat{x}_i, \ldots, x_\ell) d\mu(t)\end{aligned}$$

Of course, the formula for Δf is found by adding these two.

Remark. Harmonic forms are solutions of the optimization problem: Minimize the "Dirichlet form" $\|\delta f\|^2 + \|\partial f\|^2 =< \Delta f, f >=< \Delta^{1/2} f, \Delta^{1/2} f >$ over $f \in L^2_a(U^{\ell+1}_\alpha)$.

Remark. There is a second notion of $U^{\ell+1}_\alpha$ called the RIPS complex (see Chazal and Oudot [4]) defined by $(x_0, \ldots, x_\ell) \in U^{\ell+1}_\alpha$(RIPS) if and only if $d(x_i, x_j) \leq \alpha$ for all i, j. We have not studied a version of Theorem 3 in this case.

5 L^2 Theory of α-Harmonic 0-Forms

In this section we assume that we are in the setting of section 4, with $\ell = 0$. Thus X is a compact metric space with a probability measure and with a fixed scale $\alpha > 0$.

Recall that $f \in L^2(X)$ is α-harmonic if $\Delta_\alpha f = 0$. Moreover if $\delta : L^2(X) \to L^2_a(U^2_\alpha)$ denotes the coboundary, then $\Delta_\alpha f = 0$ if and only if $\delta f = 0$; also $\delta f(x_0, x_1) = f(x_1) - f(x_0)$ for all pairs $(x_0, x_1) \in U^2_\alpha$.

Recall that for any $x \in X$, the slice $S_{x,\alpha} = S_x \subset X^2$ is the set

$$S_x = S_{x,\alpha} = \{t \in X : \exists p \in X \text{such that } x, t \in B_\alpha(p)\}$$

. Note that $B_\alpha(x) \subset S_{x,\alpha} \subset B_{2\alpha}(x)$. It follows that $x_1 \in S_{x_0,\alpha}$ if and only if $x_0 \in S_{x_1,\alpha}$. We conclude

Proposition. *Let $f \in L^2(X)$. Then $\Delta_\alpha f = 0$ if and only if is locally constant in the sense that f is constant on $S_{x,\alpha}$ for every $x \in X$. Moreover if $\Delta_\alpha f = 0$, then*

(a) If X is connected, then f is constant.

(b) If α is greater than the maximum distance between components of X, then f is constant.

(c) For any $x \in X$, $f(x) =$average of f on $S_{x,\alpha}$ and on $B_\alpha(x)$.

(d) Harmonic functions are continuous.

We note that continuity of f follows from the fact that f is constant on each slice $S_{x,\alpha}$, and thus locally constant.

Remark. We will show that (d) is also true for harmonic 1-forms with an additional assumption on μ, (section 8) but are unable to prove it for harmonic 2-forms.

Consider next an extension of (d) to the Poisson regularity problem. If $\Delta_\alpha f = g$ is continuous, is f continuous? In general the answer is no, and we will give an example.

Note that $\Delta f = \partial\delta f$. Thus for $f \in L^2(X)$, by (4.4)

$$(*) \qquad \Delta_\alpha f(x) = 2\mu(S_{x,\alpha}) f(x) - 2\int_{S_{x,\alpha}} f(t)\, d\mu(t)$$

The following example shows that an additional assumption is needed for the Poisson regularity problem to have an affirmative solution. Let X be the closed interval $[-1, 1]$ with the usual metric d and let μ be the Lebesgue measure on X with an atom at 0, $\mu(\{0\}) = 1$. Fix any $\alpha < 1/4$. We will define a piecewise linear function on X with discontinuities at -2α and 2α as follows. Let a and b be any real numbers $a \neq b$, and define

$$f(x) = \begin{cases} \dfrac{a-b}{8\alpha} + a, & -1 \le x < -2\alpha \\ \dfrac{b-a}{4\alpha}(x - 2\alpha) + b, & -2\alpha \le x \le 2\alpha \\ \dfrac{a-b}{8\alpha} + b, & 2\alpha < x \le 1 \end{cases}$$

Using $(*)$ above one readily checks that $\Delta_\alpha f$ is continuous by computing left hand and right hand limits at $\pm 2\alpha$. (The constant values of f outside $[-2\alpha, 2\alpha]$ are chosen precisely so that the discontinuities of the two terms on the right side of $(*)$ cancel out.)

With an additional "regularity" hypothesis imposed on μ, the Poisson regularity property holds. In the rest of this section assume that $\mu(S_x \cap A)$ is a continuous function of $x \in X$ for each measurable set A. One can show that if μ is Borel regular, then this will hold provided $\mu(S_x \cap A)$ is continuous for all closed sets A (or all open sets A).

Proposition. *If $\Delta_\alpha f = g$ is continuous for $f \in L^2(X)$, f is continuous.*

Proof. From $(*)$ we have

$$f(x) = \frac{g(x)}{2\mu(S_x)} + \frac{1}{\mu(S_x)} \int_{S_x} f(t)\, d\mu(t)$$

The first term on the right is clearly continuous by our hypotheses on μ and the fact that g is continuous. It suffices to show that the function $h(x) = \int_{S_x} f(t)\, d\mu(t)$ is continuous. If $f = \chi_A$ is the characteristic function of any measurable set A, then $h(x) = \mu(S_x \cap A)$ is continuous, and therefore h is continuous for f any simple function (linear combination of characteristic functions of measurable sets). From general measure theory, if $f \in L^2(X)$, we can find a sequence of simple functions f_n such that $f_n(t) \to f(t)$ a.e, and $|f_n(t)| \leq |f(t)|$ for all $t \in X$. Thus $h_n(x) = \int_{S_x} f_n(t)\, d\mu(t)$ is continuous and

$$|h_n(x) - h(x)| \leq \int_{S_x} |f_n(t) - f(t)|\, d\mu(t) \leq \int_X |f_n(t) - f(t)|\, d\mu(t)$$

Since $|f_n - f| \to 0$ a.e, and $|f_n - f| \leq 2|f|$ with f being in $L^1(X)$, it follows from the dominated convergence theorem that $\int_X |f_n - f|\, d\mu \to 0$. Thus h_n converges uniformly to h and so continuity of h follows from continuity of h_n.

We don't have a similar result for 1-forms.

Partly to relate our framework of α-harmonic theory to some previous work, we combine the setting of section 2 with section 4. Thus we now put back the function K. Assume $K > 0$ is a symmetric and continuous function $K : X \times X \to \mathbf{R}$, and δ and ∂ are defined as in section 2, but use a similar extension to general $\alpha > 0$, of section 4, all in the L^2 theory.

Let $D : L^2(X) \to L^2(X)$ be the operator defined as multiplication by the function

$$D(x) = \int_X G(x,y)\, d\mu(y) \qquad \text{where } G(x,y) = K(x,y)\chi_{U_\alpha^2}$$

using the characteristic function $\chi_{U_\alpha^2}$ of U_α^2. So $\chi_{U_\alpha^2}(x_0, x_1) = 1$ if $(x_0, x_1) \in U_\alpha^2$ and 0 otherwise. Furthermore, let $L_G : L^2(X) \to L^2(X)$ be the integral operator defined by

$$L_G f(x) = \int_X G(x,y) f(y)\, d\mu(y)$$

Note that $L_G(1) = D$ where 1 is the constant function. When X is compact L_G is a Hilbert-Schmidt operator (this was first noted to us by Ding-Xuan Zhou). Thus L_G is trace class and self adjoint. It is not difficult to see now that $*$ takes the form

$$(**) \qquad \frac{1}{2}\Delta_\alpha f = Df - L_G f$$

For the special case $\alpha = \infty$, i.e. α is irrelevant as in section 2, this is the situation as in Smale-Zhou [19] for the case K is a reproducing kernel. As in the previous proposition

Proposition. *The Poisson Regularity Problem holds for the operator of* $**$.

To get a better understanding of $**$ it is useful to define a normalization of the kernel G and the operator L_G as follows. Let $\hat{G}: X \times X \to \mathbf{R}$ be defined by

$$\hat{G}(x,y) = \frac{G(x,y)}{(D(x)D(y))^{1/2}}$$

and $L_{\hat{G}} : L^2(X) \to L^2(X)$ be the corresponding integral operator. Then $L_{\hat{G}}$ is trace class, self adjoint, with non-negative eigenvalues, and has a complete orthonormal system of continuous eigenfunctions.

A normalized α-Laplacian may be defined on $L^2(X)$ by

$$\frac{1}{2}\hat{\Delta} = I - L_{\hat{G}}$$

so that the spectral theory of $L_{\hat{G}}$ may be transfered to $\hat{\Delta}$. (Also, one might consider $\frac{1}{2}\Delta^* = I - D^{-1}L_G$ as in Belkin, De Vito, and Rosasco [1].)

In Smale-Zhou [19], for $\alpha = \infty$, error estimates are given (reproducing kernel case) for the spectral theory of $L_{\hat{G}}$ in terms of finite dimensional approximations. See especially Belkin and Niyogi [2] for limit theorems as $\alpha \to 0$.

6 Harmonic Forms on Constant Curvature Manifolds

In this section we will give an explicit description of harmonic forms in a special case. Let X be a compact, connected, oriented manifold of dimension $n > 0$, with a Riemannian metric g of constant sectional curvature. Also, assume that g is normalized so that $\mu(X) = 1$ where μ is the measure induced by the volume form associated with g, and let d be the metric on X induced by g. Let $\alpha > 0$ be sufficiently small so that for all $p \in X$, the ball $B_{2\alpha}(p)$ is geodesically convex. That is, for $x, y \in B_{2\alpha}(p)$ there is a unique, length minimizing geodesic γ from x to y, and γ lies in $B_{2\alpha}(p)$. Note that if $(x_0, \ldots, x_n) \in U_\alpha^{n+1}$, then $d(x_i, x_j) \leq 2\alpha$ for all i, j, and thus all x_i lie in a common geodesically convex ball. Such a point defines an n-simplex with vertices $x_0, \ldots, x_n$ whose faces are totally geodesic submanifolds, which we will denote by $\sigma(x_0, \ldots, x_n)$. We will also denote the k dimensional faces by $\sigma(x_{i_0}, \ldots, x_{i_k})$ for $k < n$. Thus $\sigma(x_i, x_j)$ is the geodesic segment from x_i to x_j, $\sigma(x_i, x_j, x_k)$ is the union of geodesic segments from x_i to points on $\sigma(x_j, x_k)$ and higher dimensional simplices are defined inductively. (Since X has constant curvature, this construction is symmetric in $x_0, \ldots, x_n$.) A k dimensional face will be called degenerate if one of it's vertices is contained in one of it's $k-1$ dimensional faces.

For $(x_0, \ldots, x_n) \in U_\alpha^{n+1}$, the orientation on X induces an orientation on $\sigma(x_0, \ldots, x_n)$ (assuming it is non-degenerate). For example, if $v_1, \ldots, v_n$ denoted the tangent vectors at x_0 to the geodesics from x_0 to $x_1, \ldots, x_n$, we can define $\sigma(x_0, \ldots, x_n)$ to be positive (negative) if $\{v_1, \ldots, v_n\}$ is a positive (respectively negative) basis for the tangent space at x_0. Of course, if τ is a permutation, the orientation of $\sigma(x_0, \ldots, n)$ is equal to $(-1)^{\mathrm{sign}\tau}$ times the orientation of $\sigma(x_{\tau(0)}, \ldots, x_{\tau(n)})$. We now define $f : U_\alpha^{\ell+1} \to \mathbf{R}$ by

$$\begin{aligned} f(x_0, \ldots, x_n) &= \mu(\sigma(x_0, \ldots, x_n)) \;\text{ for } \sigma(x_0, \ldots, x_n) \text{ positive} \\ &= -\mu(\sigma(x_0, \ldots, x_n)) \;\text{ for } \sigma(x_0, \ldots, x_n) \text{ negative} \\ &= 0 \;\text{ for } \sigma(x_0, \ldots, x_n) \text{ degenerate} \end{aligned}$$

Thus f is the signed volume of oriented geodesic n-simplices. Clearly f is continuous as non-degeneracy is an open condition and the volume of a simplex varies continuously in the vertices. The main result of this section is

Theorem. *f is harmonic. In fact f is the unique harmonic n-form in $L^2_a(U^{n+1}_\alpha)$ up to scaling.*

Proof. Uniqueness follows from subsequent work with Bartholdi and Schick. We will show that $\partial f = 0$ and $\delta f = 0$. Let $(x_0, \ldots, x_{n-1}) \in U^n_\alpha$. To show $\partial f = 0$, it suffices to show, by Proposition 10, that

$$\int_{S_{x_0 \cdots x_{n-1}}} f(t, x_0, \ldots, x_{n-1})\, d\mu(t) = 0 \tag{6.1}$$

We may assume that $\sigma(x_0, \ldots, x_{n-1})$ is non-degenerate, otherwise the integrand is identically zero. Recall that $S_{x_0 \cdots x_{n-1}} = \{t \in X : (t, x_0, \ldots, x_{n-1}) \subset U^{n+1}_\alpha\} \subset B_{2\alpha}(x_0)$ where $B_{2\alpha}(x_0)$ is the geodesic ball of radius 2α centered at x_0. Let Γ be the intersection of the totally geodesic $n-1$ dimensional submanifold containing $x_0, \ldots, x_{n-1}$ with $B_{2\alpha}(x_0)$. Thus Γ divides $B_{2\alpha}(x_0)$ into two pieces B^+ and B^-. For $t \in \Gamma$, the simplex $\sigma(t, x_0, \ldots, x_{n-1})$ is degenerate and therefore the orientation is constant on each of $B+$ and B^-, and we can assume that the orientation of $\sigma(t, x_0, \ldots, x_{n-1})$ is positive on $B+$ and negative on B^-. For $x \in B_{2\alpha}(x_0)$ define $\phi(x)$ to be the reflection of x across Γ. Thus the geodesic segment from x to $\phi(x)$ intersects Γ perpendicularly at it's midpoint. Because X has constant curvature, ϕ is a local isometry and since $x_0 \in \Gamma$, $d(x, x_0) = d(\phi(x), x_0)$. Therefore $\phi : B_{2\alpha}(x_0) \to B_{2\alpha}(x_0)$ is an isometry which maps B^+ isometrically onto B^- and B^- onto B^+. Denote $S_{x_0 \cdots x_{n-1}}$ by S. It is easy to see that $\phi : S \to S$, and so defining $S^\pm = S \cap B^\pm$ it follows that $\phi : S^+ \to S^-$ and $\phi : S^- \to S^+$ are isometries. Now

$$\begin{aligned}
&\int_{S_{x_0 \cdots x_{n-1}}} f(t, x_0, \ldots, x_{n-1})\, d\mu(t) \\
&\qquad = \int_{S^+} f(t, x_0, \ldots, x_{n-1})\, d\mu(t) + \int_{S^-} f(t, x_0, \ldots, x_{n-1})\, d\mu(t) \\
&\qquad = \int_{S^+} \mu(\sigma(t, x_0, \ldots, x_{n-1}))\, d\mu(t) - \int_{S^-} \mu(\sigma(t, x_0, \ldots, x_{n-1}))\, d\mu(t)
\end{aligned}$$

Since $\mu(\sigma(t, x_0, \ldots, x_{n-1})) = \mu(\sigma(\phi(t)t, x_0, \ldots, x_{n-1}))$ for $t \in S^+$, the last two terms on the right side cancel establishing (6.1).

We now show that $\delta f = 0$. Let $(t, x_0, \ldots, x_n) \in U^{n+2}_\alpha$. Thus

$$\delta f(t, x_0, \ldots, x_n) = f(x_0, \ldots, x_n) + \sum_{i=0}^{n} (-1)^{i+1} f(t, x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

and we must show that

$$f(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i f(t, x_0, \ldots, \hat{x}_i, \ldots, x_n) \tag{6.2}$$

Without loss of generality, we will assume that $\sigma(x_0, \ldots, x_n)$ is positive. The demonstration of (6.2) depends on the location of t. Suppose that t is in the interior of the simplex $\sigma(x_0, \ldots, x_n)$. Then for each i, the orientation of $\sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ is the same as the orientation of $\sigma(x_0, \ldots, x_n)$ since t and x_i lie on the same side of the face $\sigma(x_0. \ldots, \hat{x}_i, \ldots, x_n)$, and is thus positive. On the other hand, the orientation of $\sigma(t, x_0, \ldots \hat{x}_i, \ldots, x_n)$ is $(-1)^i$ times the orientation of $\sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$. Therefore the right side of (6.2) becomes

$$\sum_{i=0}^{n} \mu(\sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n))$$

This however equals $\mu(\sigma(x_0, \ldots, x_n))$ which is the left side of (6.2), since

$$\sigma(x_0, \ldots, x_n) = \cup_{i=0}^{n} \sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$$

when t is interior to $\sigma(x_0, \ldots, x_n)$.

There are several cases when t is exterior to $\sigma(x_0, \ldots, x_n)$ (or on one of the faces), depending on which side of the various faces it lies. We just give the details of one of these, the others being similar. Simplifying notation, let F_i denote the face "opposite" x_i, $\sigma(x_0, \ldots, \hat{x}_i, \ldots, x_n)$, and suppose that t is on the opposite side of F_0 from x_0, but on the same side of F_i as x_i for $i \neq 0$. As in the above argument, the orientation of $\sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ is positive for $i \neq 0$ and is negative for $i = 0$. Therefore the right side of (6.2) is equal to

$$\sum_{i=1}^{n} \mu(\sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)) - \mu(\sigma(t, x_1, \ldots, x_n) \tag{6.3}$$

Let s be the point where the geodesic from x_0 to t intersects F_0. Then for each $i > 0$

$$\begin{aligned} \sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) = \sigma(x_0, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n) \\ \cup\, \sigma(s, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \end{aligned}$$

Taking μ of both sides and summing over i gives

$$\begin{aligned} \sum_{i=1}^{n} \mu(\sigma(x_0, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)) &= \sum_{i=1}^{n} \mu(\sigma(x_0, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)) \\ &+ \sum_{i=1}^{n} \mu(\sigma(s, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)) \end{aligned}$$

However, the first term on the right is just $\mu(\sigma(x_0, \ldots, x_n))$ and the second term is $\mu(\sigma(t, x_1, \ldots, x_n)$. Combining this with (6.3) gives us (6.2), finishing the proof of $\delta f = 0$.

Remark. The proof that $\partial f = 0$ strongly used the fact that X has constant curvature. In the case where X has variable curvature, totally geodesic n simplices don't generally exist, although geodesic triangles $\sigma(x_0, x_1, x_2)$ are well defined for $(x_0, x_1, x_2) \in U_\alpha^3$. In this case, the proof above shows that $\delta f = 0$.

7 Cohomology

Traditional cohomology theories on general spaces are typically defined in terms of limits as in Cech theory, with nerves of coverings. However, an algorithmic approach suggests a development via a scaled theory, at a given scale $\alpha > 0$. Then, as $\alpha \to 0$ one recovers the classical setting. A closely related point of view is that of persistent homology, see Edelsbrunner, Letscher, and Zomorodian [9], Zomorodian and Carlsson [25], and Carlsson [26].

We give a setting for such a scaled theory, with a fixed scaling parameter $\alpha > 0$.

Let X be a separable, complete metric space with metric d, and $\alpha > 0$ a "scale". We will define a (generally infinite) simplicial complex $C_{X,\alpha}$ associated to (X, d, α). Toward that end let $X^{\ell+1}$, for $\ell \geq 0$, be the $\ell+1$-fold Cartesian product, with metric still denoted by d, $d : X^{\ell+1} \times X^{\ell+1} \to \mathbf{R}$ where $d(x, y) = \max_{i=0,\dots,\ell} d(x_i, y_i)$. As in section 4, let

$$U_\alpha^{\ell+1}(X) = U_\alpha^{\ell+1} = \{x \in X^{\ell+1} : d(x, D_{\ell+1}) \leq \alpha\}$$

where $D_{\ell+1} \subset X^{\ell+1}$ is the diagonal, so $D_{\ell+1} = \{(t, \dots, t)\ \ell + 1 \text{ times}\}$. Then let $C_{X,\alpha} = \cup_{\ell=0}^{\infty} U_\alpha^{\ell+1}$.This has the structure of a simplicial complex whose ℓ-simplices consist of points of $U_\alpha^{\ell+1}$. This is well defined since if $x \in U_\alpha^{\ell+1}$, then $y = (x_0, \dots, \hat{x}_i, \dots, x_\ell) \in U_\alpha^\ell$, for each $i = 0, \dots, \ell$. We will write $\alpha = \infty$ to mean that $U_\alpha^\ell = X^\ell$. Following e.g. Munkres [17], there is defined a cohomology theory, simplicial cohomology, for this simplicial complex, with with cohomology vector spaces (always over $\mathbf{R}$), denoted by $H_\alpha^\ell(X)$. We especially note that $C_{X,\alpha}$ is not necessarily a finite simplicial complex. For example, if X is an open non-empty subset of Euclidean space, the vertices of $C_{X,\alpha}$ are the points of X and of course infinite in number. The complex $C_{X,\alpha}$ will be called the scaled simplicial complex, at scale α associated to X.

Example. X is finite. Fix $\alpha > 0$. In this case, for each ℓ, the set of ℓ-simplices is finite, the ℓ-chains form a finite dimensional vector space and the α-cohomology groups (i.e. vector spaces) $H_\alpha^\ell(X)$ are all finite dimensional. One can check that for $\alpha = \infty$, one has $\dim H_\alpha^0(X) = 1$ and $H_\alpha^i(X)$ are trivial for all $i > 0$. Moreover, for α sufficiently small ($\alpha < \min\{d(x, y) : x, y \in X,\ x \neq y\}$) $\dim H_\alpha^0(X)$ =cardinality of X, with $H_\alpha^i(X) = 0$ for all $i > 0$. For intermediate α, the α-cohomology can be rich in higher dimensions, but $C_{X,\alpha}$ is a finite simplicial complex.

Example. First let $A \subset \mathbf{R}^2$ be the annulus $A = \{x \in \mathbf{R}^2 : 1 \leq \|x\| \leq 2\}$. Form A^* by deleting the finite set of points with rational coordinates p/q, with $|q| \leq 10^{10}$. Then one may check that for $\alpha > 4$, $H_\alpha^\ell(A^*)$ has the cohomology of a point, for certain intermediate values of α, $H_\alpha^\ell(A^*) = H_\alpha^\ell(A)$, and for α small enough $H_\alpha^\ell(A^*)$ has enormous dimension. Thus the scale is crucial to see the features of A^* clearly.

Returning to the case of general X, note that if $0 < \beta < \alpha$ one has a natural inclusion $J : U_\beta^\ell \to U_\alpha^\ell$, $J : C_{X,\beta} \to C_{X,\alpha}$ and the restriction $J^* : L_a^2(U_\alpha^\ell) \to L_a^2(U_\beta^\ell)$ commuting with δ (a chain map).

Now assume X is compact. For fixed scale α, consider the covering $\{B_\alpha(x) : x \in X\}$, where $B_\alpha(x)$ is the ball $B_\alpha(x) = \{y \in X : d(x, y) < \alpha\}$, and the nerve of the covering is $C_{X,\alpha}$, giving the "Cech construction at scale α". Thus from the Cech cohomology theory, we may say that the limit as $\alpha \to 0$ of $H_\alpha^\ell(X) = H^\ell(X) = H_{Cech}^\ell(X)$ is the ℓ-th Cech cohomology group of X.

The next observation is to note that our construction of the scaled simplicial complex $C_{X,\alpha}$ of X follows the same path as Alexander-Spanier theory (see Spanier [21]). Thus the scaled cohomology groups $H^\ell_\alpha(X)$ will have the direct limit as $\alpha \to 0$, the Alexander-Spanier group $H^\ell_{Alex-Sp}(X)$. Thus $H^\ell(X) = H^\ell_{Alex-Sp}(X) = H^\ell_{Cech}(X)$. In fact in much of the literature this is recognized by the use of the term Alexander-Spanier-Cech cohomology. What we have done is describe a finite scale version of the classical cohomology in which the classical theory appears as a limit.

Now that we have defined the scaled cohomology groups, scale α, $H^\ell_\alpha(X)$ for a metric space X, our Hodge theory suggests this modification. From Theorem 3, we have considered instead of arbitrary cochains (i.e. arbitrary functions on $U^{\ell+1}_\alpha$ which give the definition here of $H^\ell_\alpha(X)$), cochains defined by L^2 functions on $U^{\ell+1}_\alpha$. Thus we have constructed cohomology groups at scale α from L^2 functions on $U^{\ell+1}_\alpha$, $H^\ell_{\alpha,L^2}(X)$, when $\alpha > 0$, and X is a metric space equipped with Borel probability measure.

Cohomology Identification Problem (CIP). To what extent are $H^\ell_{L^2,\alpha}(X)$and $H^\ell_\alpha(X)$ isomorphic?

This is important via Theorem 3 which asserts that $H^\ell_{\alpha,L^2}(X) \to \mathrm{Harm}^\ell_\alpha(X)$ is an isomorphism, in case $H^\ell_{\alpha,L^2}(X)$ is finite dimensional.

One may replace L^2 functions in the construction of the α-scale cohomology theory by continuous functions. As in the L^2 theory, this gives rise to cohomology groups $H^\ell_{\alpha,cont}(X)$. Analagous to CIP we have the simple question: To what extent is the natural map $H^\ell_{\alpha,cont}(X) \to H^\ell_\alpha(X)$ is an isomorphism?

Note that in the case X is finite, or $\alpha = \infty$, we have an affirmative answer to this question, as well as CIP (see sections 2 and 3).

Proposition A. *There is a natural injective linear map* $Harm^\ell_{cont,\alpha}(X) \to H^\ell_{cont,\alpha}(X)$.

Proof. The inclusion, which is injective

$$J : \mathrm{Im}_{cont,\alpha}\delta \oplus \mathrm{Harm}^\ell_{cont,\alpha}(X) \to \mathrm{Ker}_{cont,\alpha}$$

induces an injection

$$J^* : \mathrm{Harm}^\ell_{cont,\alpha}(X) = \frac{\mathrm{Im}_{cont,\alpha}\delta \oplus \mathrm{Harm}^\ell_{cont,\alpha}(X)}{\mathrm{Im}_{cont,\alpha}\delta} \to \frac{\mathrm{Ker}_{cont,\alpha}}{\mathrm{Im}_{cont,\alpha}} = H^\ell_{cont,\alpha}(X)$$

and the proposition follows.

8 Continuous Hodge Theory on the Neighborhood of the Diagonal

As in the last section, (X,d) will denote a compact metric space equipped with a Borel probability measure μ. For topological reasons (see section 6) it would be nice to have a Hodge decomposition for continuous functions on $U^{\ell+1}_\alpha$, analogous to the continuous theory on the whole space (section 4). We will use the following notation. $C^{\ell+1}_\alpha$ will denote the continuous alternating real valued functions on $U^{\ell+1}_\alpha$, $\mathrm{Ker}_{\alpha,cont}\Delta_\ell$ will denote the functions in $C^{\ell+1}_\alpha$ that are harmonic, and $\mathrm{Ker}_{\alpha,cont}\delta_\ell$ will denote those elements of $C^{\ell+1}_\alpha$ that are closed. Also, $H^\ell_{\alpha,cont}(X)$ will denote the quotient space (cohomology space) $\mathrm{Ker}_{\alpha,cont}\delta_\ell/\delta(C^\ell_\alpha)$ We raise the following question, analogous to Thereom 3.

Question (Continuous Hodge Decomposition). *Under what conditions on X, and $\alpha > 0$ is it true that there is the following orthogonal (with respect to the L^2 inner product) direct sum decomposition*

$$C_\alpha^{\ell+1} = \delta(C_\alpha^\ell) \oplus \partial(C_\alpha^{\ell+2}) \oplus Ker_{\alpha,cont}\Delta_\ell$$

where $Ker_{cont,\alpha}\Delta_\ell$ is isomorphic to $H^\ell_{\alpha,cont}(X)$, with every element in $H^\ell_{\alpha,cont}(X)$ having a unique representative in $Ker_{\alpha,cont}\Delta_\ell$?

There is a related analytical problem that is analogous to elliptic regularity for partial differential equations, and in fact elliptic regularity features prominently in classical Hodge theory.

The Poisson Regularity Problem. *For $\alpha > 0$, and $\ell > 0$, suppose that $\Delta f = g$ where $g \in C_\alpha^{\ell+1}$ and $f \in L^2_a(U_\alpha^{\ell+1})$. Under what conditions on (X, d, μ) is f continuous?*

Theorem. *An affirmative answer to the Poisson Regularity problem, together with closed image $\delta(L^2_a(U^\ell_\alpha))$ implies an affirmative solution to the continuous Hodge decomposition question.*

Proof. Assume that the Poisson regularity property holds, and let $f \in C_\alpha^{\ell+1}$. From theorem 3 we have the L^2 Hodge decomposition

$$f = \delta f_1 + \partial f_2 + f_3$$

where $f_1 \in L^2_a(U^\ell_\alpha)$, $f_2 \in L^2_a(U^{\ell+2}_\alpha)$ and $f_3 \in L^2_a(U^{\ell+1}_\alpha)$ with $\Delta f_3 = 0$. It suffices to show that f_1 and f_2 can be taken to be continuous, and f_3 is continuous. Since $\Delta f_3 = 0$ is continuous, f_3 is continuous by Poisson regularity. We will show that $\partial f_2 = \partial(\delta h_2)$ where δh_2 is continuous (and thus f_2 can be taken to be continuous). Recall (corollary of the Hodge Lemma in section 2) that the following maps are isomorphisms

$$\delta : \partial(L^2_a(U^{\ell+2}_\alpha)) \to \delta(L^2_a(U^{\ell+1}_\alpha)) \text{ and } \partial : \delta(L^2_a(U^{\ell}_\alpha)) \to \partial(L^2_a(U^{\ell+1}_\alpha))$$

For all $\ell \geq 0$. Thus

$$\partial f_2 = \partial(\delta h_2) \text{ for some } h_2 \in L^2_a(U^{\ell+1}_\alpha)$$

Now,

$$(*) \qquad \Delta(\delta(h_2)) = \delta(\partial(\delta(h_2))) + \partial(\delta(\delta(h_2))) = \delta(\partial(\delta(h_2))) = \delta(\partial(f_2))$$

since $\delta^2 = 0$. However, from the decomposition for f we have, since $\delta f_3 = 0$

$$\delta f = \delta(\partial f_2)$$

and since f is continuous δf is continuous, and therefore $\delta(\partial f_2)$ is continuous. It then follows from Poisson regularity and $*$ that δh_2 is continuous as to be shown. A dual argument shows that $\delta f_1 = \delta(\partial h_1)$ where ∂h_1 is continuous, completing the proof.

Notice that a somewhat weaker result than Poisson regularity would imply that f_3 above is continuous, namely regularity of harmonic functions.

Harmonic Regularity Problem. *For $\alpha > 0$, and $\ell > 0$, suppose that $\Delta f = 0$ where $f \in L^2_a(U^{\ell+1}_\alpha)$. What conditions on (X, d, μ) would imply f is continuous?*

Under some additional conditions on the measure, we have answered this for $\ell = 0$ (see section 5) and can do so for $\ell = 1$, which we now consider. We will first derive an expression for a harmonic 1-form f in terms measures of slices and integrals of f over subsets of X^2. Let $f \in L^2_a(U^2_\alpha)$ be harmonic. Then from Proposition 10, section 4, since $\partial f = 0$ we have for $x \in X$

$$\int_{S_x} f(t, x)\, d\mu(t) = 0 \tag{8.1}$$

Since $\delta f = 0$, we have

$$f(x_0, x_1) = f(x_0, s) - f(x_1, s) \tag{8.2}$$

for all $(x_0, x_1, s) \in U^3_\alpha$ or equivalently, $(x_0, x_1) \in U^2_\alpha$ and $s \in S_{x_0 x_1}$. Integrating (8.2) over $s \in S_{x_0 x_1}$ gives

$$f(x_0, x_1) = \frac{1}{\mu(S_{x_0 x_1})} \left(\int_{S_{x_0 x_1}} f(x_0, s)\, d\mu(s) - \int_{S_{x_0 x_1}} f(x_1, s)\, d\mu(s) \right) \tag{8.3}$$

We now use (8.2) to extend f to a somewhat larger set. Rewriting (8.2), note that

$$f(x_0, s) = f(x_0, x_1) - f(s, x_1)$$

for $s \in S_{x_0 x_1}$ and $(x_0, x_1) \in U^2_\alpha$. However, the right side is actually defined whenever $s \in S_{x_1}$. Therefore, the above equation defines a unique extension of f to $\{(x_0, s) : s \in S_{x_1}$whenever $(x_0, x_1) \in U^2_\alpha\}$ such that $\delta f = 0$. Clearly this extension is in L^2, since the right side of the above equation is in L^2. Now, we integrate (8.2) over $t \in S_{x_0}$, with s in place of x_1 and t in place of s to get

$$\mu(S_{x_0}) f(x_0, s) = \int_{S_{x_0}} f(t, s)\, d\mu(t)$$

A similar computation gives

$$\mu(S_{x_1}) f(x_1, s) = \int_{S_{x_1}} f(u, s)\, d\mu(u)$$

and substituting these into (8.3) yields

$$\begin{aligned} f(x_0, x_1) = \frac{1}{\mu(S_{x_0 x_1})} \Bigg(& \frac{1}{\mu(S_{x_0})} \int_{S_{x_0 x_1}} \int_{S_{x_0}} f(t, s)\, d\mu(t) d\mu(s) \\ & - \frac{1}{\mu(S_{x_1})} \int_{S_{x_0 x_1}} \int_{S_{x_1}} f(u, s)\, d\mu(u) d\mu(s) \Bigg) \end{aligned} \tag{8.4}$$

Note that the variables x_0 and x_1 occur only within the measure, and domains of integration on the right. These terms will be continuous under regularity assumptions on μ analogous to the Poisson regularity proposition for 0-forms in section 5. Thus from (8.4) we have the following proposition.

Proposition. *Assume that $\mu(S_x)$, $\mu(S_{xy})$ and $\mu^2((S_{xy} \times S_x) \cap A)$ are continuous for $x, y \in X$ and all A measurable. If f is an α-harmonic 1-form in $L^2_a(U^2_\alpha)$, then f is continuous.*

As in section 5, if μ is Borel regular, it suffices that the hypotheses hold for all A closed (or all A open).

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