A Union of Euclidean Spaces is Euclidean

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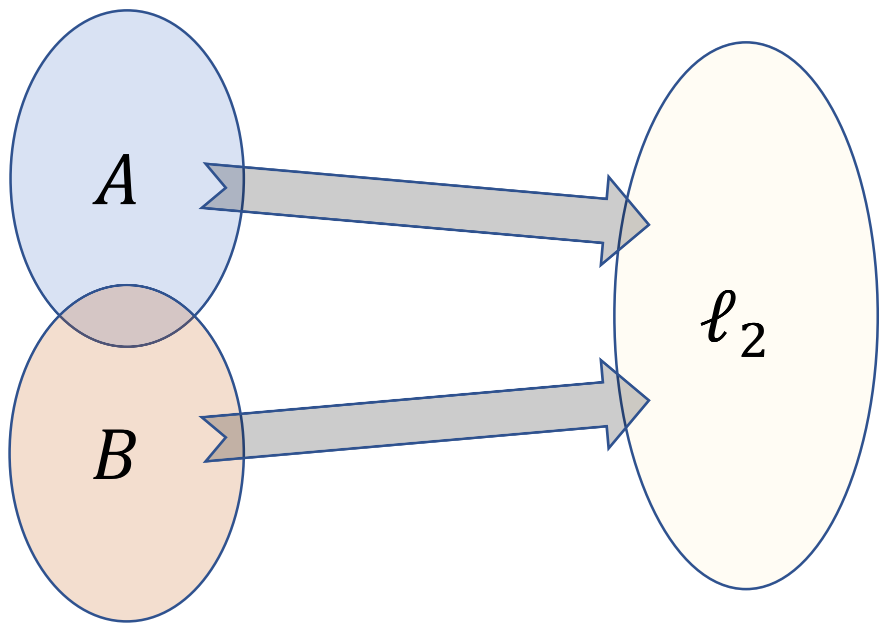
Problem by Assaf Naor

Suppose that metric space (X, d) is a union of two metric spaces A and B that isometrically embed into ℓ_2. Does X necessarily embed into ℓ_2 with a constant distortion?



Motivation

The problem is closely connected to research in theoretical computer science on “local-global properties” of metric spaces [Arora, Lovász, Newman, Rabani, Rabinovich, Vempala `06; Charikar, M, Makarychev `07]

Why are computer scientists interested?

Results imply strong lower bounds for Sherali-Adams linear programming relaxations for many combinatorial optimization problems, including Sparsest Cut, Vertex Cover, Max Cut, Unique Games. [Charikar, M, Makarychev `09]

Our Results

Q: Suppose that metric space (X, d) is a union of two metric spaces A and B that embed isometrically into ℓ_2. Does X necessarily embed into ℓ_2 with a constant distortion?

A: Yes, X embeds into ℓ_2 with distortion at most 8.93.

$A \hookrightarrow \ell_2^a$ with distortion α, $B \hookrightarrow \ell_2^b$ with distortion β

$$\Downarrow$$

$X = A \cup B \hookrightarrow \ell_2^{a+b+1}$ with distortion at most $11\alpha\beta$

Approach

This talk: consider the isometric case.

$$\varphi_1 : A \hookrightarrow \ell_2$$
$$\varphi_2 : B \hookrightarrow \ell_2$$

We will define 3 maps:

- $\bar{\varphi}_1 : A \cup B \hookrightarrow \ell_2$, a 7-Lipschitz extension of φ_1 to X
- $\bar{\varphi}_2 : A \cup B \hookrightarrow \ell_2$, a 7-Lipschitz extension of φ_2 to X
- $\Delta(x) = d(x, A) - d(x, B)$

$$\psi = \bar{\varphi}_1 \oplus \bar{\varphi}_2 \oplus \Delta$$

Approach

$$\psi = \bar{\varphi}_1 \bigoplus \bar{\varphi}_2 \bigoplus \Delta$$

Assume that we have

- $\bar{\varphi}_1 : A \cup B \hookrightarrow \ell_2$, a 7-Lipschitz extension of φ_1 to X
- $\bar{\varphi}_2 : A \cup B \hookrightarrow \ell_2$, a 7-Lipschitz extension of φ_2 to X
- $\Delta(x) = d(x, A) - d(x, B)$

First,

$$\|\psi\|_{Lip} = \|\bar{\varphi}_1 \bigoplus \bar{\varphi}_2 \bigoplus \Delta\|_{Lip} \leq \sqrt{7^2 + 7^2 + 2^2}$$

since $\|\Delta\|_{Lip} \leq 2$.

Approach

$$\psi = \bar{\varphi}_1 \oplus \bar{\varphi}_2 \oplus \Delta$$

- $\bar{\varphi}_1$ ensures that distances between points in A don't decrease:

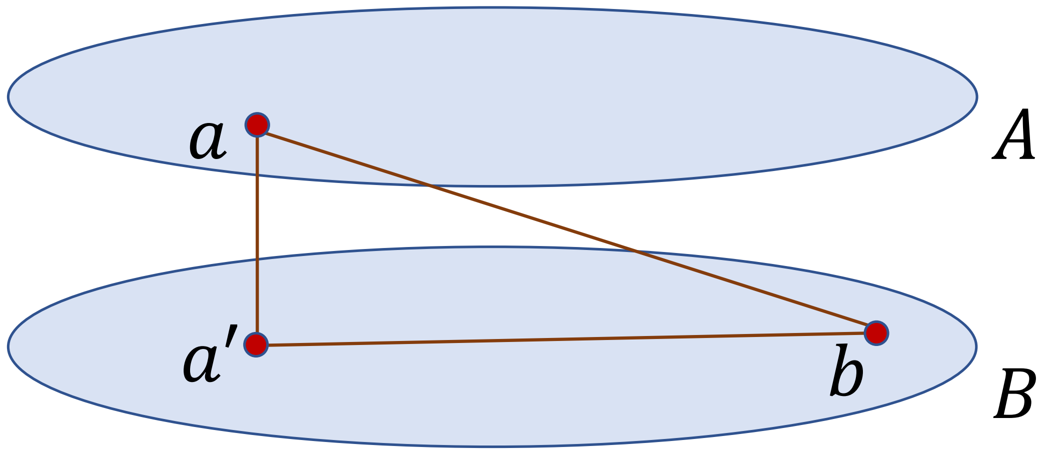
  $\bar{\varphi}_1|_A = \varphi_1$ is an isometric embedding of A into ℓ_2.

- $\bar{\varphi}_2$ ensures that distances between points in B don't decrease.

- Δ ensures that distances between points $a \in A$ and $b \in B$ don't decrease by more than a constant factor.

Approach

$$\psi = \bar{\varphi}_1 \oplus \bar{\varphi}_2 \oplus \Delta$$



If $d(a, a') \ll d(a, b)$ then

$$\|\bar{\varphi}_2(a) - \bar{\varphi}_2(b)\| \approx \|\bar{\varphi}_2(a') - \bar{\varphi}_2(b)\| = d(a', b) \approx d(a, b)$$

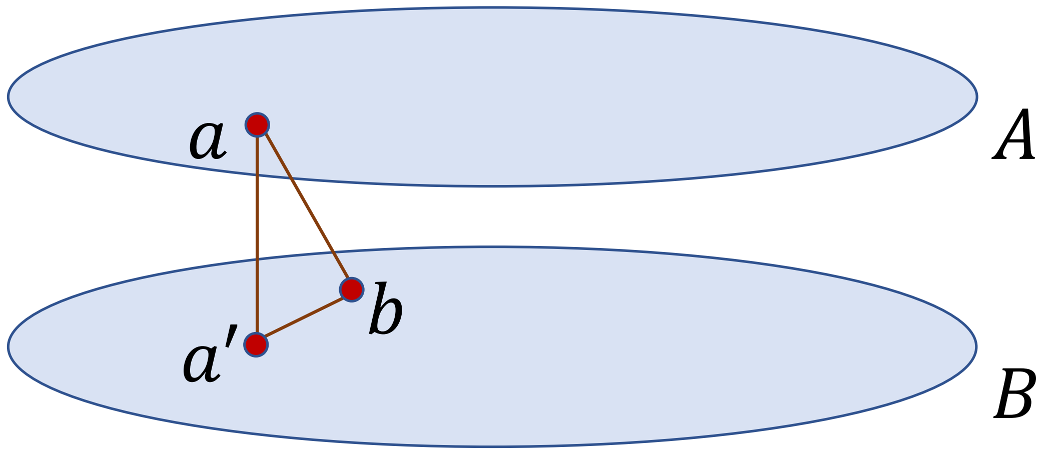
If $d(a, a') \approx d(a, b)$ then

$$\|\Delta(a) - \Delta(b)\| \geq d(a, a') \approx d(a, b)$$

a' is the closest point to a in B

Approach

$$\psi = \bar{\varphi}_1 \oplus \bar{\varphi}_2 \oplus \Delta$$



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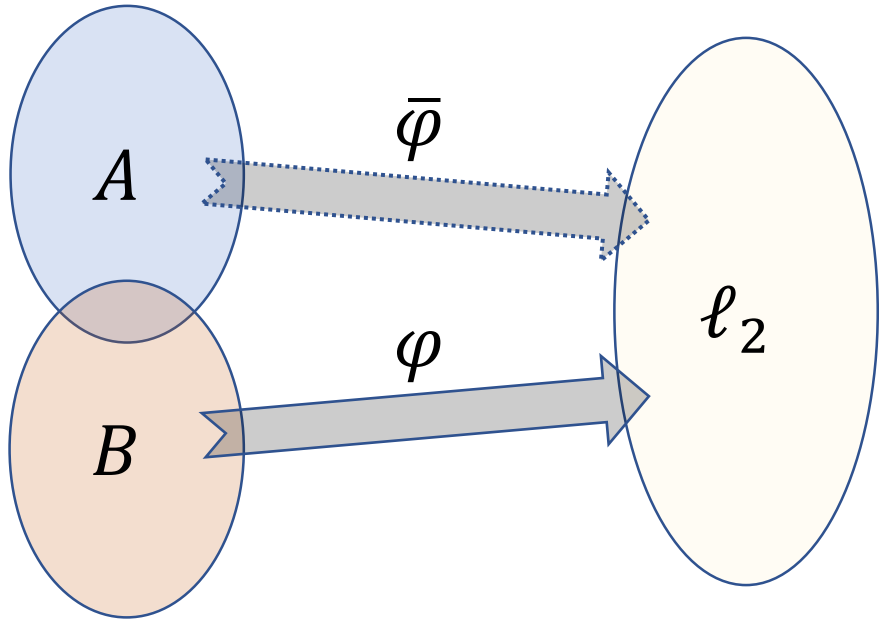
Constructing maps $\bar{\varphi}_1$ and $\bar{\varphi}_2$

Goal:

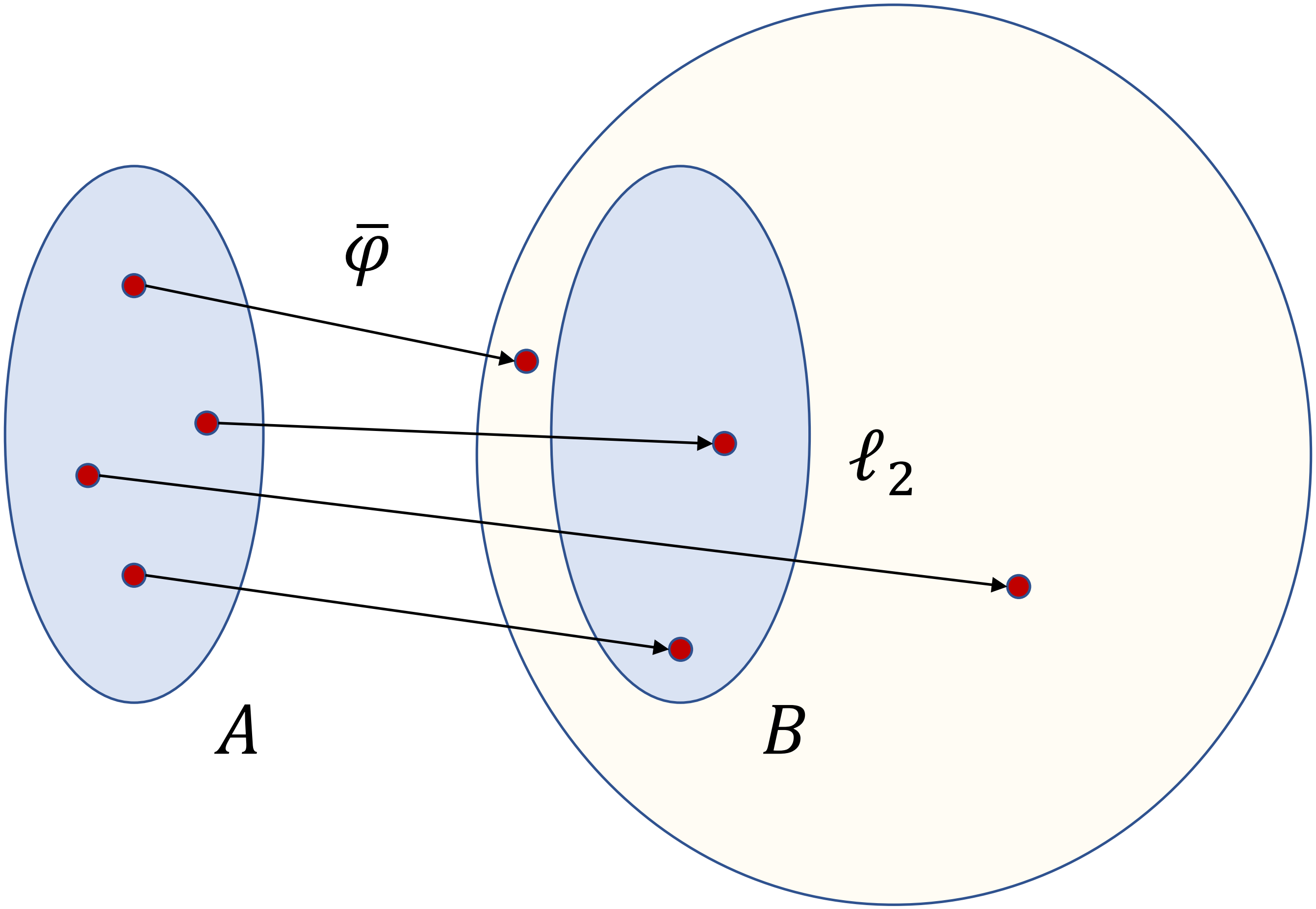
Given a map $\varphi \equiv \varphi_2 : B \to \ell_2$

find a Lipschitz extension $\bar{\varphi} : A \cup B \to \ell_2$ of φ.



Constructing maps $\bar{\varphi}_1$ and $\bar{\varphi}_2$

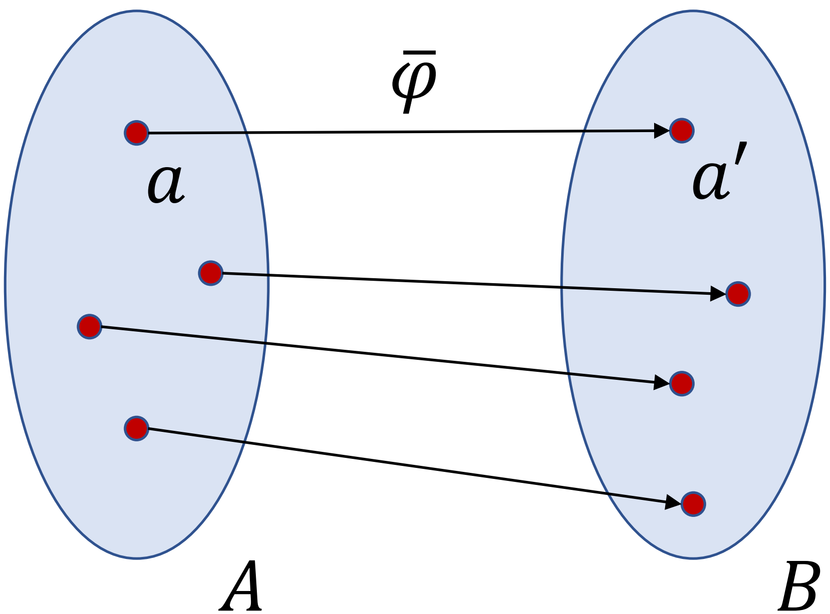
Assume that $B \subset \ell_2$ and $\varphi = id$; $|A \cup B| < \infty$.



Constructing map $\bar{\varphi}$

Idea 1: map every a to the closest $a' \in B$ w.r.t. d.

Issue: the map may not be Lipschitz.

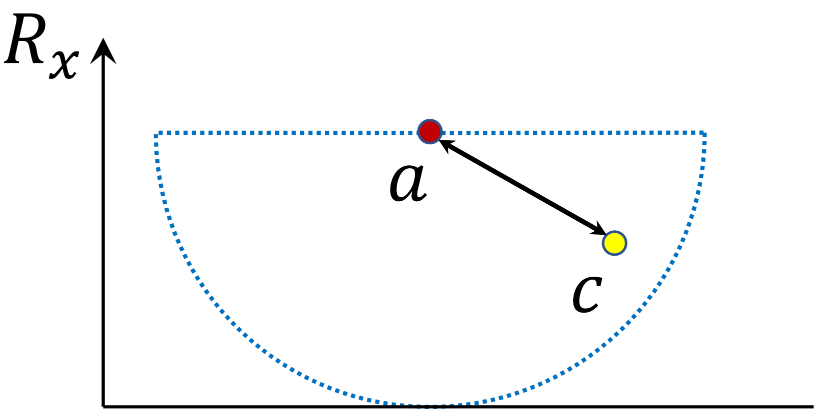


Cover for A

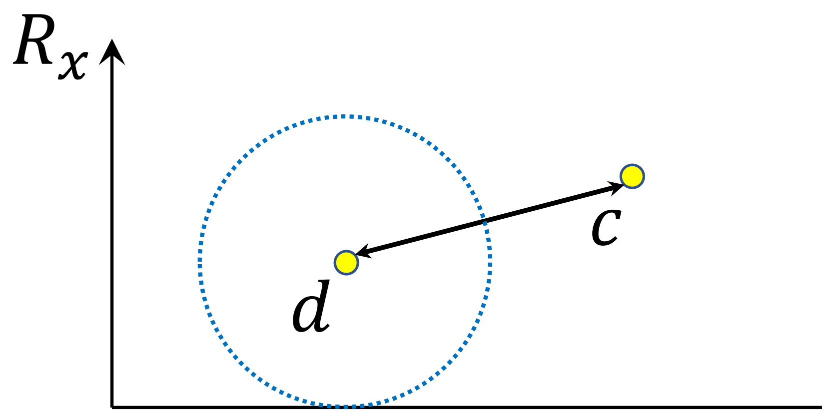
Let $R_a = d(a, B)$ for $a \in A$.

$C \subset A$ is a cover for A if

- for every $a \in A$, there is $c \in C$ s.t.
$$d(a, c) \leq R_a \text{ and } R_c \leq R_a$$
- for every $c, d \in C$: $d(c, d) \geq \min(R_c, R_d)$.



$a \in A$ is close to some $c \in C$



points in C are "separated"

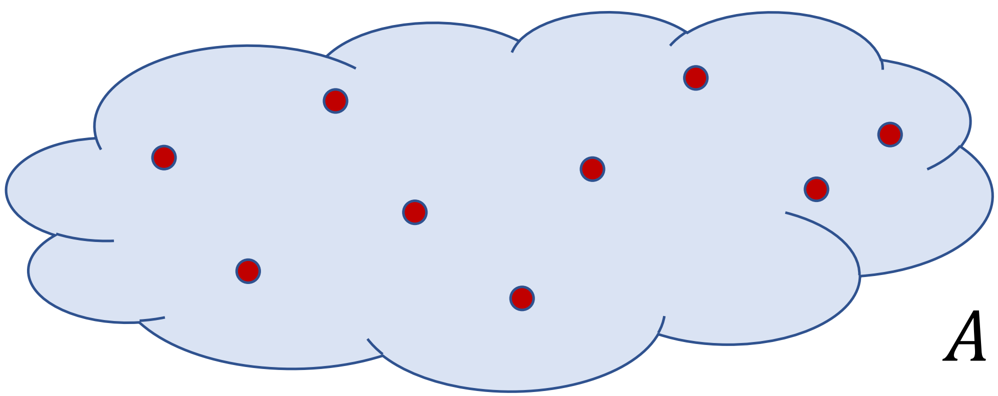
Cover for A

Prove by induction that there is always a cover C.

Let $c \in A$ be the point in A with the least value of R_c.

By induction, there is a cover C' for $A \setminus \mathrm{Ball}(c, R_c)$.

Let $C = C' \cup \{c\}$.



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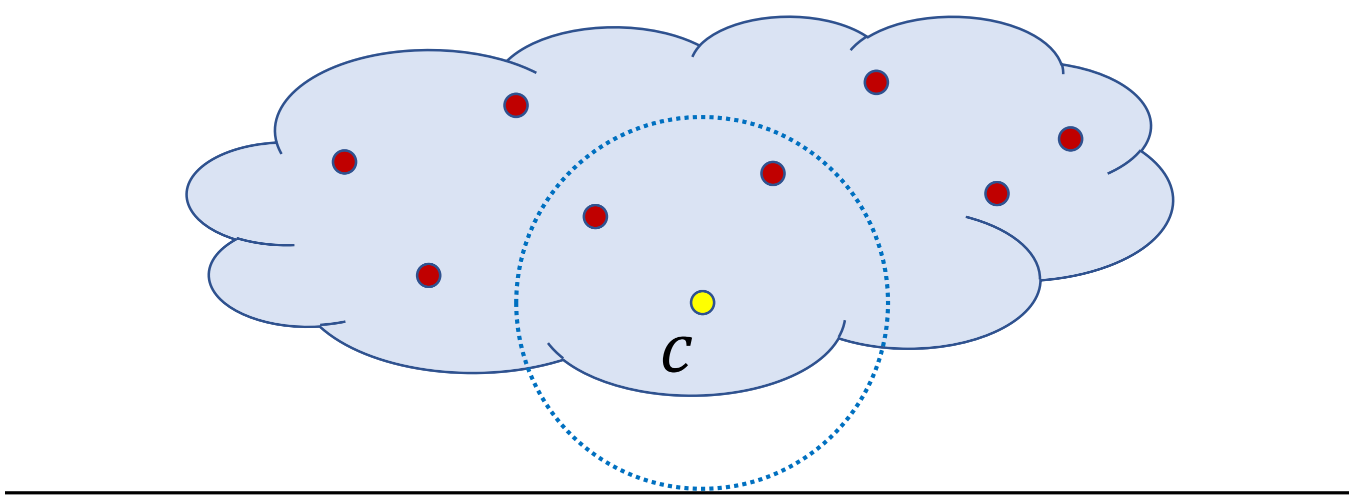

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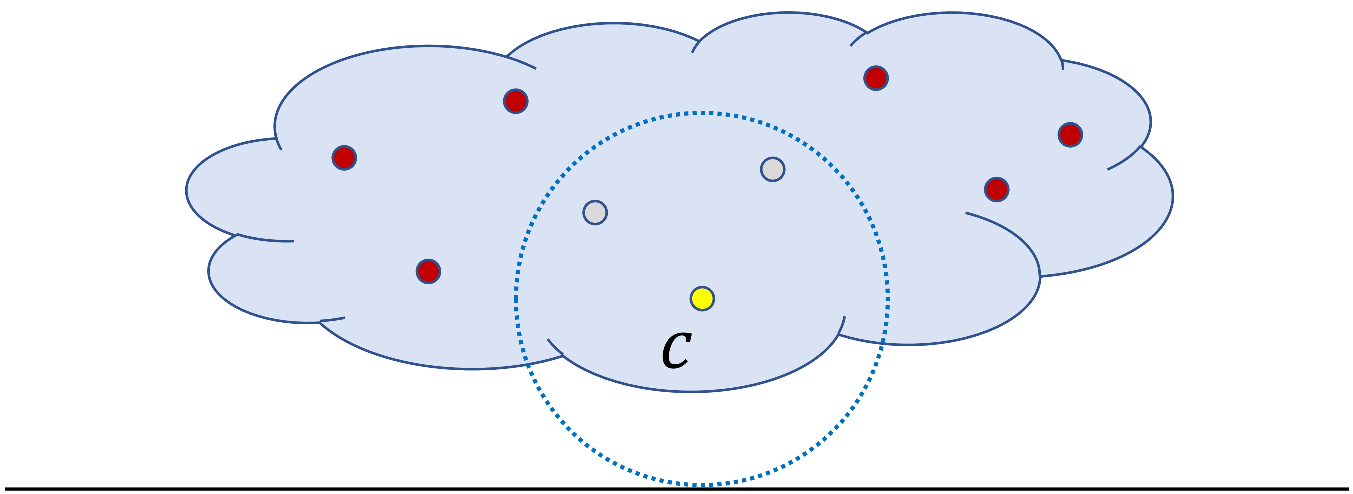

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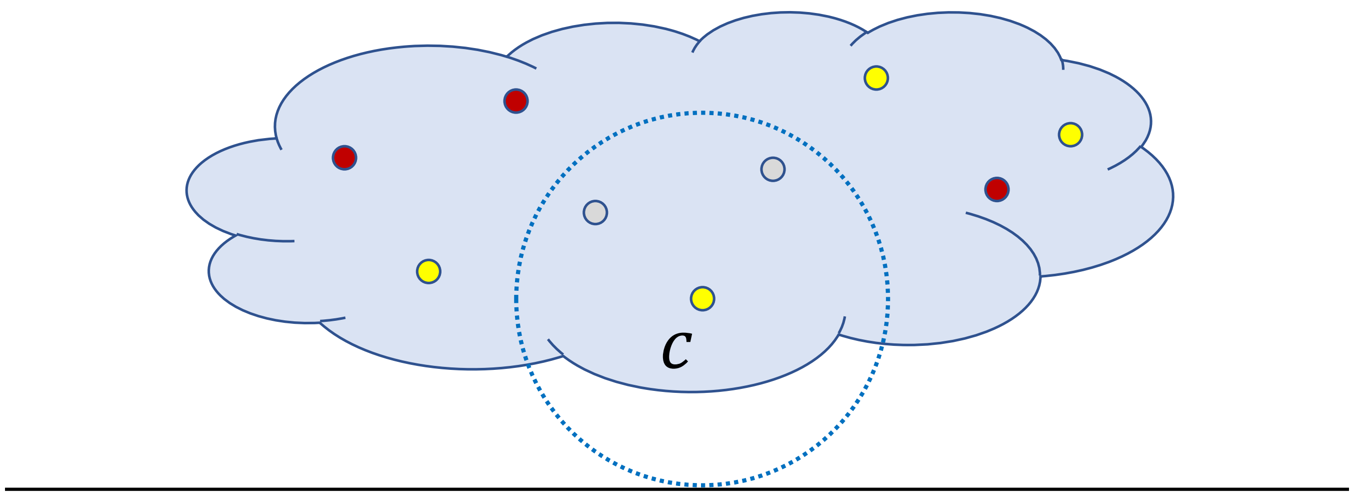
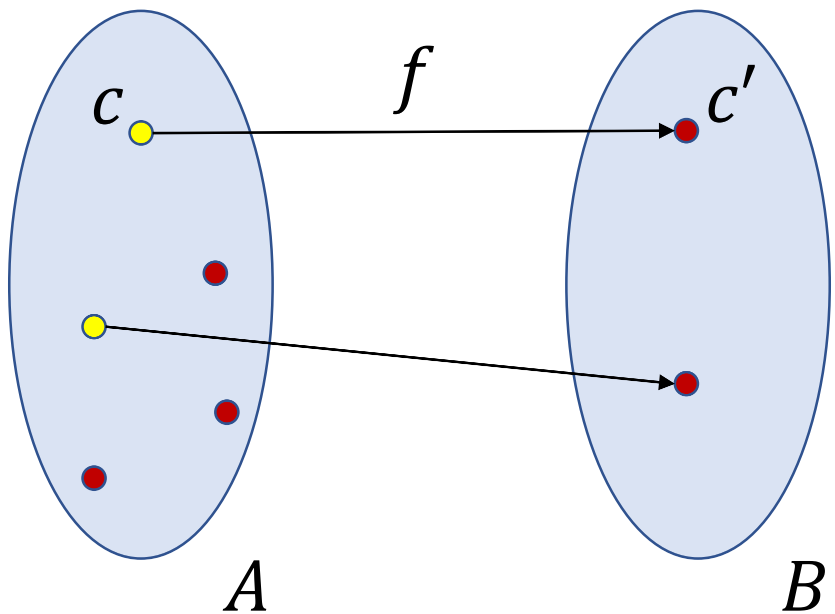


Constructing map $\bar{\varphi}$

Idea 2: map every $c \in C$ to the closest $c' \in B$.

The map is 4-Lipschitz.



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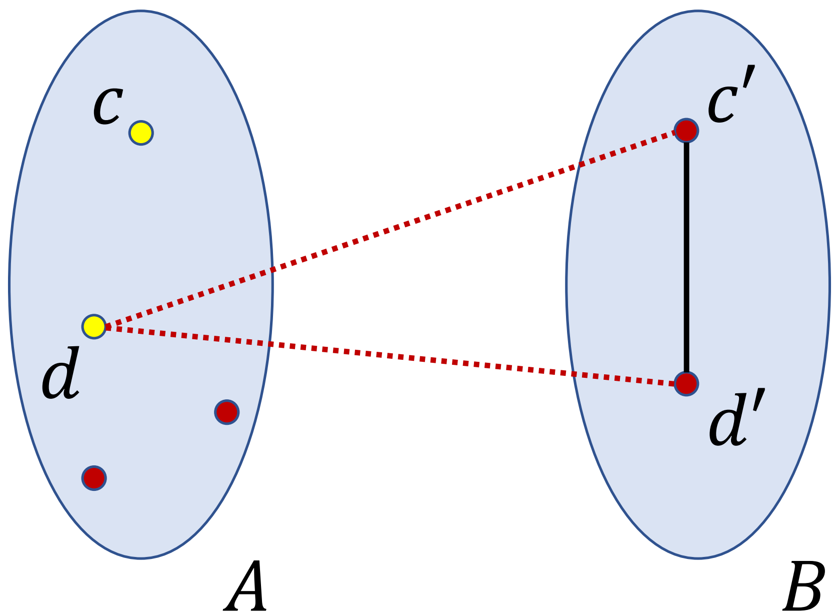

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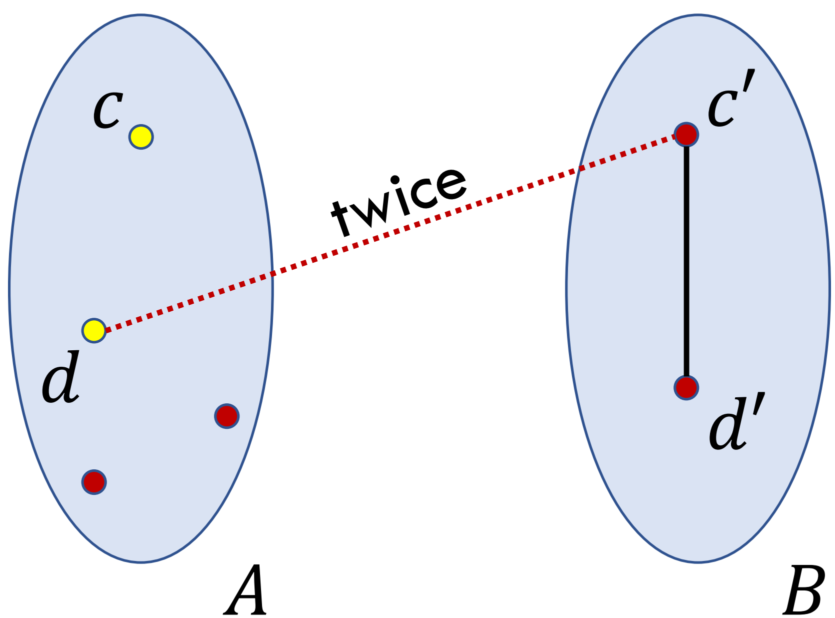

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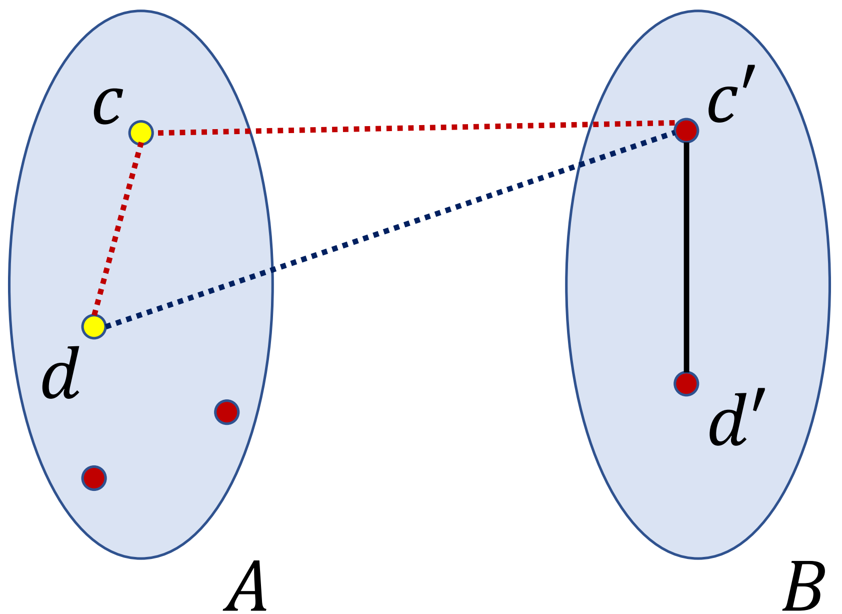
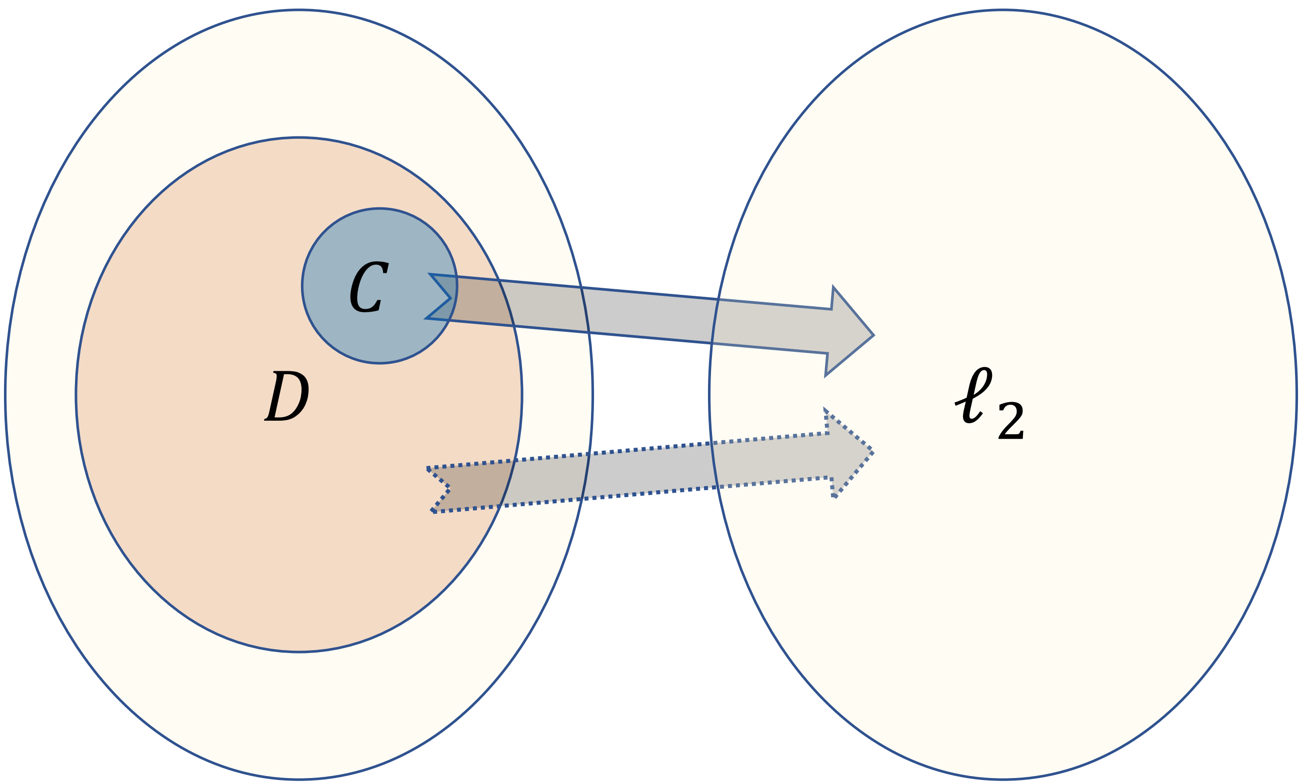


$$d(c', d') \leq 2\, d(c, d) + 2\, d(c, c') \leq 4 d(c, d)$$

Kirszbraun Theorem

Let $C \subset D \subset \ell_2$ and f be a Lipschitz map from C to ℓ_2. There exists an extension $g: D \to \ell_2$ of f such

$$\|g\|_{Lip} = \|f\|_{Lip}$$



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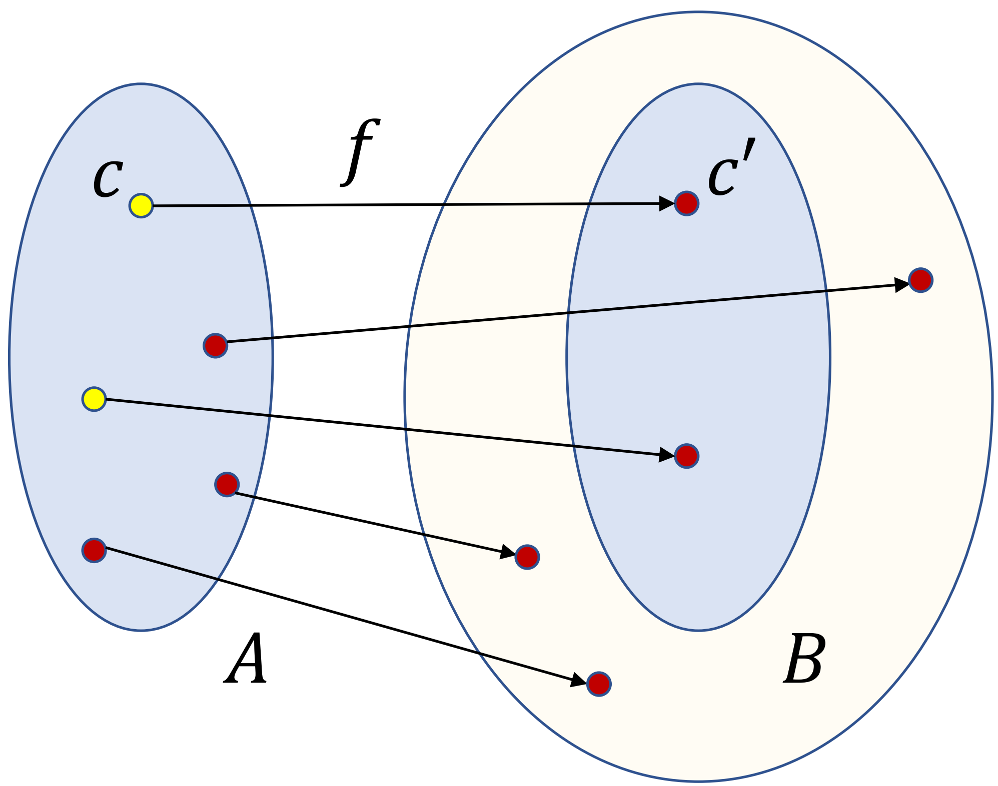
Idea 2: map every $c \in C$ to the closest $c' \in B$.

Extend f from C to A using the Kirszbraun theorem.



Constructing map $\bar{\varphi}$

Idea 2: map every $c \in C$ to the closest $c' \in B$.

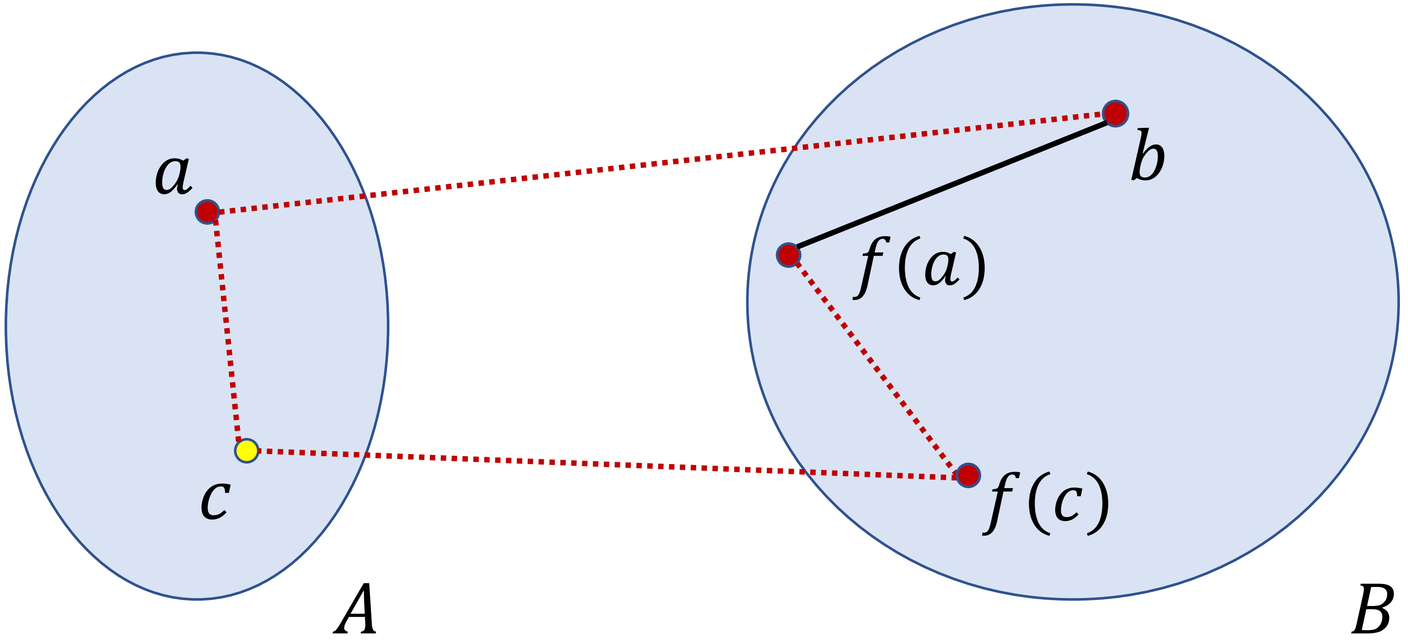
Extend f from C to A using the Kirszbraun theorem.

$$\bar{\varphi}(u) = \begin{cases} f(u), & \text{if } u \in A \\ u, & \text{if } u \in B \end{cases}$$

$\bar{\varphi}(u)$ is 7-Lipschitz:

- $\bar{\varphi}|_A$ is 4-Lipschitz
- $\bar{\varphi}|_B$ is 1-Lipschitz
- $\|\bar{\varphi}(a) - \bar{\varphi}(b)\| = \|f(a) - b\| \leq \cdots$

Constructing map $\bar{\varphi}$



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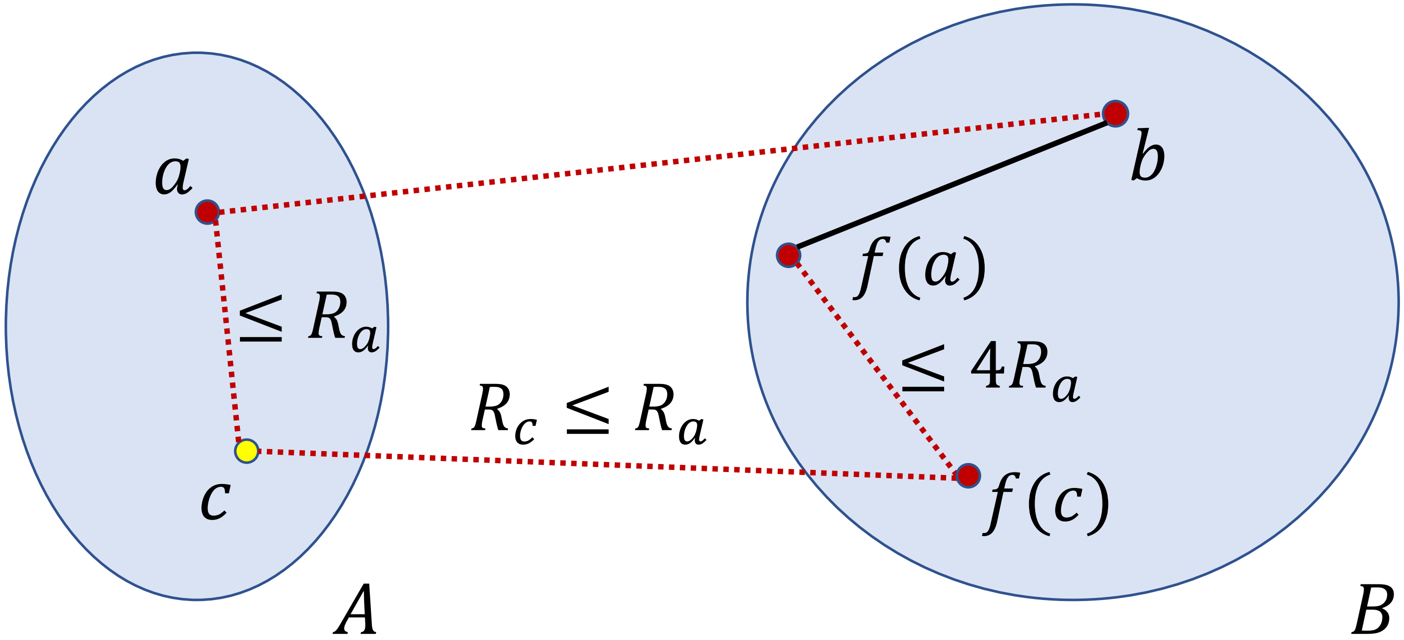
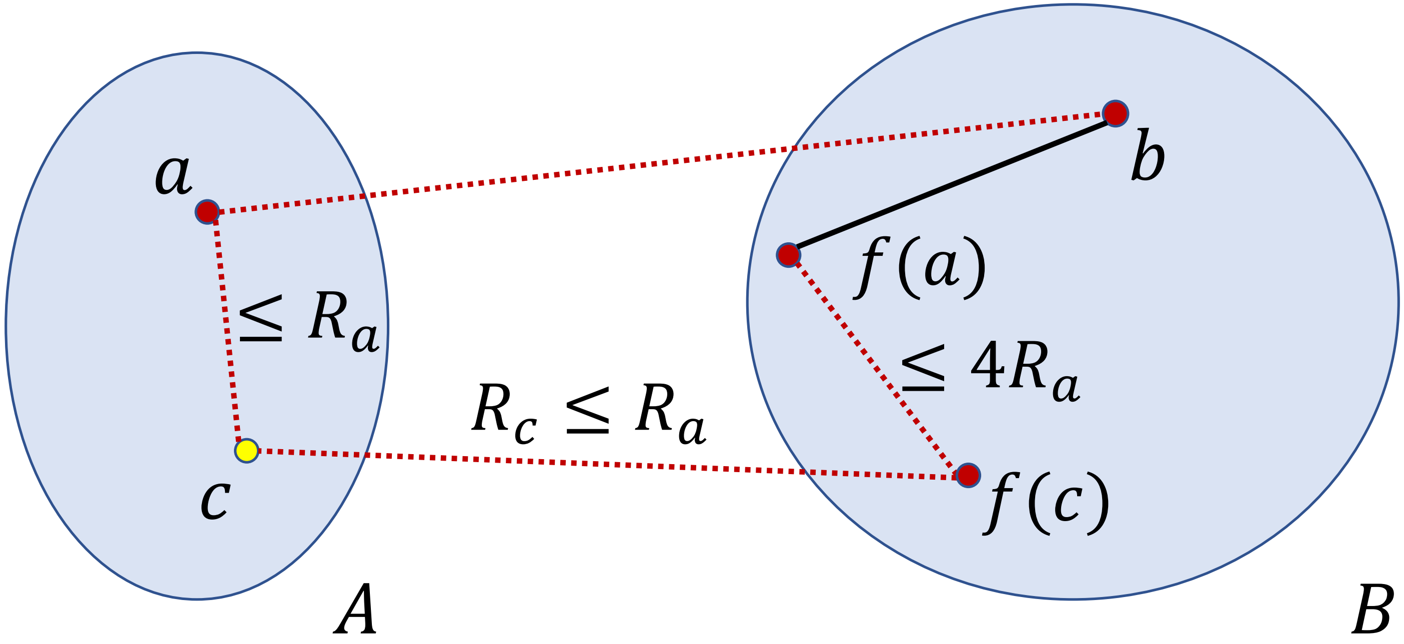


$$\|f(a) - b\| \leq 6R_a + d(a,b) \leq 7d(a,b)$$

Constructing map $\bar{\varphi}$



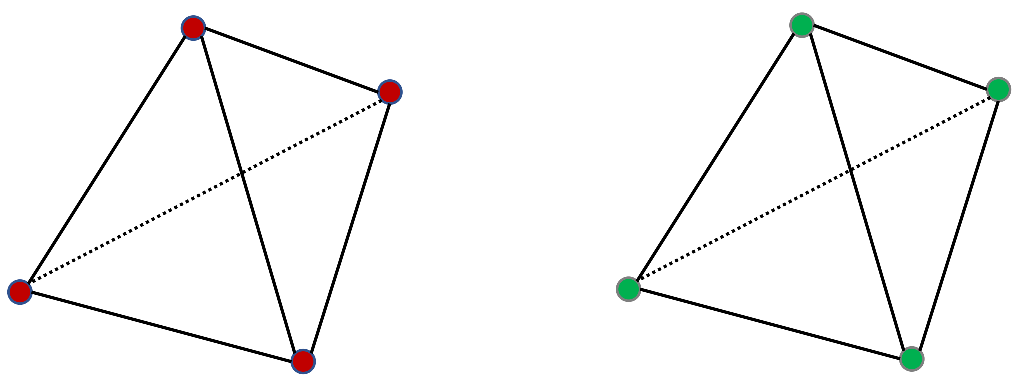
$$\|f(a) - b\| \leq 6R_a + d(a,b) \leq 7d(a,b)$$

Q.E.D.

Lower Bound

There exists a metric space $X = A \cup B$ s.t.

- A and B isometrically embed into ℓ_2
- every embedding of X into ℓ_2 has distortion at least $3 - \varepsilon_n$, where $n = |A| = |B|$ and $\varepsilon_n \to 0$ as $n \to \infty$



Open Problems

1. Find the least value of D s.t. if $A, B \hookrightarrow \ell_2$ isometrically, then $A \cup B \hookrightarrow \ell_2$ with distortion at most D. We know that $D \in [3, 8.93)$.
2. Study the problem for other ℓ_p. We conjecture that the answer is negative for every $p \notin \{2, \infty\}$.
3. What happens if $X = A_1 \cup \cdots \cup A_k$ and each $A_i \hookrightarrow \ell_2$ isometrically? We only know that $c \log k \le D \le 2^{Ck}$.
4. Assume that every subset of X of size $\sqrt{|X|}$ isometrically embeds into ℓ_2. What is the least distortion with which $X \hookrightarrow \ell_2$?

More results and open problems in the paper!