Question 0.1. Given a set of $n$ jobs, where job $i$ has processing time $p_i > 0$ and start time $s_i$, and only one job can run on one machine at a given time. Determine the minimum number of machines required to schedule all the jobs and also the schedule (i.e. the map from jobs to machines).

This is the example on pg. 122 of the Kleinberg & Tardos book.

Question 0.2. Given a set of $n$ jobs, where job $i$ has processing time $p_i > 0$ and weight $w_i > 0$, design an algorithm which outputs a schedule (i.e. a map from jobs to start times) which minimizes $\sum_{i=1}^{n} w_i C_i$ ($C_i$ is the actual completion time for job $i$ in your schedule). Note that here you have only a single machine.

This is problem 13 on pg. 194 of the Kleinberg & Tardos book. As a hint try proving correctness/optimality using an exchange argument similar to the example on pg. 128.
Here is a sketch of the solution for qn 0.2.

Algorithm: Schedule the jobs in decreasing order of \( \frac{w_i}{p_i} \) (ties broken arbitrarily).

Proof of optimality. Let \( R \) denote the schedule output by the algorithm above. W.l.o.g. we can assume that \( R_k = k \) i.e. the \( k \)th job scheduled by \( R \) is labeled \( k \). Suppose that \( O \) is some optimal schedule which differs from \( R \) for the first time at position \( k \) i.e. say \( O_k = j \) and \( O_i = R_i \) for \( i < k \). Construct a new schedule \( P^k \) such that \( P^k_i := O_i (= R_i) \) for \( i < k \), \( P^k_k := k \), and \( P^k_i := O_{i-1} \) for \( i > k \) and \( O_{i-1} \neq k \) (i.e. make sure not to schedule job \( k \) twice).

Lemma 0.3.

\[
\sum_{i=1}^{n} w_i C_i^{P^k} \leq \sum_{i=1}^{n} w_i C_i^{O}.
\]

Proof. Note that translating the completion times \( C_i^{P^k} \) and \( C_i^{O} \) by a constant amount does not change the direction of the inequality. So let the \( t = 0 \) be the completion time of job \( k - 1 \) (in both schedules). Let \( O_m = k \) and \( T := \sum_{i=k+1}^{m} p_{O_i} \). Observe that

\[
\sum_{i} w_i C_i^{P^k} - \sum_{i} w_i C_i^{O} = -w_k(T + p_j) + p_k \sum_{i=k}^{m} w_{O_i}.
\]

Since \( \frac{w_k}{p_k} \geq \frac{w_{O_i}}{p_{O_i}} \) for \( i \geq k \) we have \( w_k(T + p_j) \geq p_k \sum_{i=k}^{m} w_{O_i} \). Hence the lemma follows.

Observe that by construction \( P^k_i = R_i \) for \( i \leq k \). Repeated application of the lemma above allows us to construct optimal schedules \( P^k \) for all \( k \leq n \), thus the schedule \( R \) costs no more than some optimal schedule \( O \). Therefore the optimality of \( R \) follows.