

Homework 3

Due: Thursday, March 12 in class.

**General remark:** whenever you are asked to provide an  $\alpha$ -approximation algorithm, you need to prove that your algorithm outputs a feasible  $\alpha$ -approximate solution.

1. **Multicut**

- (a) Prove that the undirected Multicut problem is at least as hard to approximate as Vertex Cover, even when the input graph is a tree of height 1 (a star).
- (b) Prove that the flow-cut gap for the Multicut problem on trees is 2.

2. **Congestion Minimization** Show that the integrality gap of the LP-relaxation of directed Congestion Minimization is at least  $\Omega(\log n / \log \log n)$ , where  $n$  is the number of graph vertices.

Hint: Use the construction we used in class to lower bound the integrality gap for Machine Minimization.

3. **EDP** Show exact polynomial time algorithm for the special case of the Maximum Edge Disjoint Paths problem, where the input graph is a tree. (Notice that we assume that all edge capacities are 1).

Hint: Use dynamic programming.

4. **Multiway Cut** Consider the following generalization of the Multiway Cut problem. We are given an undirected graph  $G = (V, E)$  with costs  $w_e \geq 0$  on edges, and a subset  $T = \{t_1, \dots, t_k\}$  of vertices called terminals. Additionally, each vertex  $v \in V$  has a list  $L(v) \subseteq T$  of *forbidden* terminals. The goal is to partition the graph vertices into  $k$  subsets  $C_1, \dots, C_k$ , such that for all  $i : 1 \leq i \leq k, t_i \in C_i$ , and each vertex  $v \in V$  belongs to some cluster  $C_j$  with  $j \notin L(v)$ .

- (a) Extend the LP relaxation for the multiway cut problem to this generalized setting.
- (b) Consider the following LP-rounding algorithm. At the beginning of the algorithm, all the vertices are unassigned. While there is an unassigned vertex, perform the following procedure:

Procedure Partition

For  $i = 1, 2, \dots, k$ :

- Choose  $r_i \in (0, 1)$  uniformly at random.
- For each unassigned vertex  $v$  with  $x_i(v) \geq r_i$ , assign  $v$  to  $C_i$ .

- i. Prove that the above algorithm produces a feasible solution for the generalized multiway cut problem.
  - ii. Prove that the algorithm achieves an  $O(\log n)$ -approximation.  
Hint: Bound the number of calls to Procedure Partition and bound the expected cost of each execution of the procedure.
- (c) Design a factor  $4/3$ -approximation algorithm for generalized multiway cut with  $k = 3$  terminals. (Partial credit will be given for constant factor approximation algorithm).