Homework 4

Due: Thursday, March 19, 3pm. No submissions will be accepted after the deadline.

1. Recall the definitions of bipartite and non-bipartite sparsest cut. In both problems the input is a graph $G = (V, E)$ with costs $c_e \geq 0$ on edges, and a set $\{(s_1, t_1), \ldots, (s_k, t_k)\}$ of source-destination pairs.

   In *bipartite* sparsest cut, the goal is to find a partition $(S, \overline{S})$ of vertices, minimizing the ratio $c(E(S, \overline{S}))/|D(S, \overline{S})|$, where $E(S, \overline{S}) = \left\{(u, v) \in E \mid u \in S, v \in \overline{S}\right\}$ and $D(S, \overline{S}) = \left\{(s_i, t_i) \mid s_i \in S, t_i \in \overline{S}\right\}$.

   In *non-bipartite* sparsest cut, the goal is to find a subset $E' \subseteq E$ of edges minimizing the ratio of $c(E')$ to the number of source-destination pairs (s_i, t_i) disconnected by E'. We say that pair (s_i, t_i) is disconnected by E' iff the graph induced by $E \setminus E'$ does not contain any paths connecting s_i to t_i.

   Given an instance G of the sparsest cut problem, let $\mathsf{OPT}_{BP}(G)$ and $\mathsf{OPT}_{NBP}(G)$ denote the costs of the optimal bipartite and non-bipartite sparsest cuts, respectively.

   (a) Show that for any undirected graph G, $\mathsf{OPT}_{BP}(G) = \mathsf{OPT}_{NBP}(G)$.
   (Partial credit will be given for showing that $\mathsf{OPT}_{BP}(G)$ and $\mathsf{OPT}_{NBP}(G)$ are within factor 2 from each other.)

   (b) Show that for any directed graph G, $\mathsf{OPT}_{NBP}(G) \leq \mathsf{OPT}_{BP}(G)$.

   (c) Show that for directed graphs, the ratio $\mathsf{OPT}_{BP}(G)/\mathsf{OPT}_{NBP}(G)$ can be unbounded.

2. (a) Show a lower bound of $\Omega(\log n)$ on the integrality gap of the LP relaxation for undirected sparsest cut studied in class.
   Hint: use the same construction as for minimum multicut.

   (b) Prove that Bourgain's theorem is essentially tight, by showing that any embedding of general metrics into ℓ_1 must incur $\Omega(\log n)$ distortion.
   Hint: Show that existence of a better embedding would contradict (a).