Symbolic Computation Jan 11, 2010

Lecture 4: Vector Space and Inner Product

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let $a : R$ in which R denotes the type of the field of all the real numbers, an abstract structure of a vector space $\mathcal{V}$ can be defined by a quadruple $< V, F, \cdot, + >$, where V denotes a set, F denotes a field, $\cdot$ denotes a type operator that $\cdot : F \times V \rightarrow V$, and $+$ denotes another type operator that $+ : V \times V \rightarrow V$. Please note that we will use symbols like x, y, z etc. for the elements in V, and a, b, c etc. for those in F.

For example, let $x_1, ..., x_d : V$, and suppose $x_1, ..., x_d$ are linearly independent, we have $\forall y : V$, $y = a_1 \cdot x_1 + ... + a_d \cdot x_d$.

One most often used set of V is R^d (read as "d fold cartesian of R"). Due to the existence of zero element in F, a vector space always contains a zero vector. We can intuitively define the concept of orthogonality: two vectors a_1 and a_2 are orthogonal in the case when $a_1 \cdot a_2 = 0$, i.e. the inner product of a_1 and a_2 is zero. For example, considering $a_1, a_2 : R^2$, let $a_1 = (\theta_1, \theta_2)$ and $a_1 = (\tau_1, \tau_2)$, we have $\theta_1 \cdot \tau_1 + \theta_2 \cdot \tau_2 = 0$.

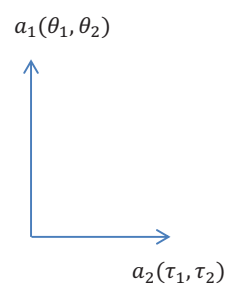

Figure 1: orthogonal

Note: In type theory, there is no well-formed mathematical representation for inner product so far.

Linear Transformation $x \rightarrow \rho(x)$: A linear transformation between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ is a map $\rho : \mathcal{V} \rightarrow \mathcal{W}$ such that the following hold:

1. $\rho(v_1 + v_2) = \rho(v_1) + \rho(v_2)$ for any vectors v_1 and v_2 in V, and
2. $\rho(\alpha v) = \alpha \rho(v)$ for any scalar α. For example:

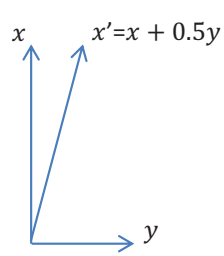

Figure 2: instance of linear transformation

Let's define $\mathfrak{F}(\mathcal{V})$, which yields inner product on $\mathcal{V} :< V, F, \cdot, + >$. Suppose $\mathcal{V}$ is isomorphic to $\mathcal{U}$ (i.e., if $x + y = z$ in $\mathcal{V}$, then $\rho(x) + \rho(y) = \rho(z)$ in $\mathcal{U}$), we have that $\mathfrak{F}(\mathcal{V})$ is isomorphic to $\mathfrak{F}(\mathcal{U})$.

Claim: any two d-dimensional vector spaces are isomorphic, i.e. R^d.

There is no natural basis for vector space, i.e. all basis are equivalent.

In type theory, vector space is presented as

$$\exists \alpha \left(\begin{array}{c} \cdot : \beta \times \alpha \to \alpha \\ + : \alpha \times \alpha \to \alpha \end{array} \right) \tag{1}$$