1. Give a derivation of the following sequent using the axioms in the first installment of the notes (A1 through A21).

\[ \emptyset ; \text{even} = (\lambda (x : \text{int}) \exists y : \text{int} \ x = y + y) \vdash \text{even} : \text{int} \rightarrow \text{Boole} \]

Your derivation should be a numbered sequence of lines where each line is justified by an axiom and a specification of which preceding lines are used as antecedents.

2. Define a function if : \text{Boole} \times \text{int} \times \text{int} \rightarrow \text{int} with the property that if(\Phi, n, m) equals n if \Phi is true and m otherwise. Your definition should be well formed under axioms A1 through A21. More specifically, it should be given as if = e and the axioms should allow you to derive \emptyset ; if = e \vdash if : \text{Boole} \times \text{int} \times \text{int} \rightarrow \text{int}. You do not have to give the derivation.

3. Give a definition of the factorial function of the form fact = e which is well formed under the axioms A1 through A21. We adopt the convention that fact(n) = 1 for n ≤ 0. For your definition e give a derivation of the following sequent.

\[ \emptyset ; \text{fact} = e \vdash \text{fact} : \text{int} \rightarrow \text{int} \]

4. Recursive definitions are usually phrased formally in terms of a fixed point operator as in the following definition of factorial.

\[ \text{fact} = \text{fix}((\lambda (f : \text{int} \rightarrow \text{int}) (\lambda (x : \text{int}) \text{if}(x \leq 1, 1, x * f(x - 1)))))) \]

Give a well-formed definition of the function fix such that this definition of fact is both well formed and correct and also works for other “recursive definitions”. More specifically, let F be the argument to fix in the above expression. Note that F(fact) = fact. The function fact is the unique fixed point of F. The function fix should return the fixed point of its argument F whenever F has a unique fixed point. You do not have to give the well-formedness derivation of your definition.