PS4 Solutions

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Problem 1

Existence

For any $v \in V$, let $\sum_{i=1}^{d} \alpha_i(v) x_i$ be its expansion in the $x_1, \ldots, x_d$ basis. We must show that these function $\alpha_i : V \to \mathbb{R}$ are linear (hence are elements of V^*) and form a basis for V^*. First, note that $\alpha_i(x_i) = 1$ and $\alpha_i(x_j) = 0$ for $j \neq i$, from inspection of the basis expansions for x_i and x_j (which are x_i and x_j themselves, respectively).

To see that these functions are linear, consider $v, w \in V$ and $c \in \mathbb{R}$. Then we may express the basis expansion for $cv + w$ in terms of the basis expansions for v and w as:

$$\begin{aligned} cv + w &= c\sum_{i=1}^{d} \alpha_i(v) x_i + \sum_{i=1}^{d} \alpha_i(w) x_i \\ &= \sum_{i=1}^{d} (c\alpha_i(v) + \alpha_i(w)) x_i \end{aligned}$$

Since basis expansions are unique, this implies that $\alpha_i(cv + w) = c\alpha_i(v) + \alpha_i(w)$ for all i. Hence, $\alpha_i \in V^*$. Since there are d such α_is, showing that they constitute a basis requires only proving that they are linearly independent. Suppose not–that is, suppose that there exists a set of coefficients $\beta_i \in \mathbb{R}$ which are not all zero such that:

$$0 = \sum_{i=1}^{d} \beta_i \alpha_i$$

Letting j be chosen such that $\beta_j \neq 0$:

$$\alpha_j = \sum_{i \neq j} \frac{\beta_i}{\beta_j} \alpha_i$$

Since $\alpha_j(x_j) = 1$, and $\alpha_i(x_j) = 0$ for all $i \neq j$, it follows that:

$$1 = 0$$

This is a contradiction. Hence, $\alpha_1, \ldots, \alpha_d$ must be a basis for V^*.

Another way of showing linear independence would be to consider an arbitrary $f \in V^*$, and noting that by the linearity of f, and taking the basis expansion of its argument:

$$\begin{aligned} f(v) &= f\left(\sum_{i=1}^{d} \alpha_i(v) x_i\right) \\ &= \sum_{i=1}^{d} f(x_i) \alpha_i(v) \end{aligned}$$

Which shows that f may be represented as a linear combination of the elements α_i, with coefficients $f(x_i)$. Since this is true for any $f \in V^*$, it shows that $\alpha_1, \ldots, \alpha_d$ spans V^*, and therefore is a basis.

Uniqueness

Now suppose that $\beta_1, \ldots, \beta_d$ is a basis for V^* such that $\beta_i(x_i) = 1$ and $\beta_i(x_j) = 0$ for all $i \neq j$.

Since $\alpha_i \in V^*$ and $\beta_1, \ldots \beta_d$ are a basis for V^*, it follows that there exist coefficients $c_{i,1}, \ldots, c_{i,d}$ such that $\alpha_i = \sum_{j=1}^{d} c_{i,j}\beta_j$ for all i.

By the above, we have that $v = \sum_{i=1}^{d} \alpha_i(v) x_i$ for every $v \in V$. Hence, by the above:

$$\alpha_i(v) = \sum_{j=1}^{d} c_{i,j}\beta_j \left(\sum_{k=1}^{d} \alpha_k(v) x_k \right)$$

By the linearity of β_j:

$$\alpha_i(v) = \sum_{k=1}^{d} \alpha_k(v) \sum_{j=1}^{d} c_{i,j}\beta_j(x_k)$$

Since $\beta_j(x_k) = \delta_{jk}$, where δ is the Kronecker delta function:

$$\alpha_i(v) = \sum_{j=1}^{d} c_{i,j}\alpha_j(v)$$

Since this holds for all $v \in V$:

$$\alpha_i = \sum_{j=1}^{d} c_{i,j}\alpha_j$$

Finally, since $\alpha_1, \ldots, \alpha_d$ is a basis for V^*, the basis expansion of α_i must be unique, which implies that $c_{i,j} = \delta_{ij}$ for all i, j. By the definition of $c_{i,j}$:

$$\begin{aligned} \alpha_i &= \sum_{j=1}^{d} c_{i,j}\beta_j \\ \alpha_i &= \beta_i \end{aligned}$$

Which shows that the basis $\alpha_1, \ldots, \alpha_d$ is unique.

Same holds in reverse

By the above, there is a unique basis $\hat{x}_1, \ldots, \hat{x}_d$ for V^{**} which satisfies $\hat{x}_i(\alpha_i) = 1$ and $\hat{x}_i(\alpha_j) = 0$ if $i \neq j$. Mapping this basis under the natural isomorphism between V^{**} and V from the previous homework gives the desired result.

Problem 2

I will use the notation of the above solution to problem 1, rather than the notation in the problem set. For any $v \in V$ and $\beta \in V^*$, we may take the basis expansion of v:

$$\beta(v) = \beta\left(\sum_{i=1}^{d} \alpha_i(v) x_i \right)$$

By the linearity of β:

$$\beta(v) = \sum_{i=1}^{d} \alpha_i(v)\beta(x_i)$$

Since $\alpha_1, \ldots, \alpha_d$ is a basis for V^*, we may write $\beta = \sum_{i=1}^{d} c_i \alpha_i$:

$$\beta(v) = \sum_{i=1}^{d} \alpha_i(v) \sum_{j=1}^{d} c_j \alpha_j(x_i)$$

Since $\alpha_j(x_i) = \delta_{ij}$:

$$\beta(v) = \sum_{i=1}^{d} \alpha_i(v) c_i$$

This is the desired result.

Problem 3

For any $v, w \in V$, we may take the basis expansions of v and w:

$$vMw = \left(\sum_{i=1}^{d} \alpha_i(v) x_i \right) M \left(\sum_{j=1}^{d} \alpha_j(w) x_j \right)$$

By the fact that M is bilinear:

$$vMw = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i(v) \alpha_j(w) (x_i M x_j)$$

This is the desired result.

Problem 4

By problem 3, we may write:

$$vMw = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i(v) \alpha_j(w) (x_i M x_j)$$

For $x_1, \ldots, x_d$ a basis for V. We will prove existence of such a basis (though not a well-typed expression for it!), by orthogonalizing an existing basis. Accordingly, let $y_1, \ldots, y_d$ be a basis for V. I will now perform the Graham-Schmidt orthonormalization process on $y_1, \ldots, y_d$ to yield an orthonormal (relative to M) basis $x_1, \ldots, x_d$. Define $z_1 = y_1$, and define $z_i = y_i - \sum_{j=1}^{i-1} \frac{y_i M z_j}{z_j M z_j} z_j$ for all $i > 1$. There are two things to note about this definition: first, that each z_i is defined as a linear combination of $y_1, \ldots, y_i$, with the coefficient on y_i being 1. Since the y_is are linearly independent, this implies that the z_is are linearly independent, and hence they constitute a basis. Also, note that $z_j M z_j > 0$ since $z_j \neq 0$, so the division in the definition of z_i cannot cause any problems. To continue, I claim that $z_i M z_j = 0$ for all $j < i$. This will be a proof by induction. For $i = 1$, the claim holds trivially. Now suppose by the induction that $z_j M z_k = 0$ for all $k < j < i$, and consider the following for $k < i$:

$$z_i M z_k = \left(y_i M z_k - \sum_{j=1}^{i-1} \frac{y_i M z_j}{z_j M z_j} z_j M z_k \right)$$

By the induction, $z_j M z_k = 0$ if $j \neq k$ on the far RHS (inside the sum) above:

$$\begin{aligned} z_i M z_k &= (y_i M z_k - y_i M z_k) \\ &= 0 \end{aligned}$$

This completes the induction, and proves that $z_1, \ldots, z_d$ are orthogonal relative to M. Finally, we define $x_i = \frac{z_i}{\sqrt{z_i M z_i}}$. Once again, since $z_i M z_i > 0$, the square-root, and the division, are perfectly safe. Since this is nothing

but a scaling of the orthogonal (relative to M) basis z_i, we must have that the x_is form an orthogonal (relative to M) basis, and furthermore that since $x_i M x_i = \frac{z_i M z_i}{\left(\sqrt{z_i M z_i}\right)^2} = 1$, that $x_1, \ldots, x_d$ in fact form an *orthonormal* basis relative to M.

Now, for any $v, w \in V$, if we expand v and w in the basis $x_1, \ldots, x_d$:

$$vMw = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i(v)\, \alpha_j(w)\, (x_i M x_j)$$

Since $x_i M x_j = \delta_{ij}$:

$$vMw = \sum_{i=1}^{d} \alpha_i(v)\, \alpha_i(w)$$

This is the desired result. Note that unlike in the previous problems, we did *not* find a well-typed expression for this basis–we only proved its existence.

Problem 5

There are a huge number of possible answers to this problem. Here's one. Consider the following:

$$f = \lambda v : V.\text{the}\,(w : V \text{ s.t. } Ww = Qv)$$

Since W is an inner product (in particular, it is of full-rank), it is invertible, showing that there is indeed a unique w satisfying the above, for every v. Hence, the above expression is well-typed. To see that this is linear, we need only note that for all $u, v : V$ and $c : \mathbb{R}$:

$$\begin{aligned} Wf(cv + u) &= Q(cv + u) \\ &= cQv + Qu \\ &= cWf(v) + Wf(u) \end{aligned}$$

And once-more using the fact that W is of full-rank, the above shows that $f(cv + u) = cf(v) = f(u)$. Since $Wf = Q$, we see that we may interpret $f : V \to V$ as $W^{-1}Q$, where $W^{-1} : V^* \to V$ and $W, Q : V \to V^*$.