Problem 1

Existence

For any \( v \in V \), let \( \sum_{i=1}^{d} \alpha_{i}(v) x_{i} \) be its expansion in the \( x_{1}, \ldots, x_{d} \) basis. We must show that these function \( \alpha_{i} : V \rightarrow \mathbb{R} \) are linear (hence are elements of \( V^{*} \)) and form a basis for \( V^{*} \). First, note that \( \alpha_{i}(x_{i}) = 1 \) and \( \alpha_{i}(x_{j}) = 0 \) for \( j \neq i \), from inspection of the basis expansions for \( x_{i} \) and \( x_{j} \) (which are \( x_{i} \) and \( x_{j} \) themselves, respectively).

To see that these functions are linear, consider \( v, w \in V \) and \( c \in \mathbb{R} \). Then we may express the basis expansion for \( cv + w \) in terms of the basis expansions for \( v \) and \( w \) as:

\[
(cv + w) = c \sum_{i=1}^{d} \alpha_{i}(v) x_{i} + \sum_{i=1}^{d} \alpha_{i}(w) x_{i}
\]

\[
= \sum_{i=1}^{d} (c \alpha_{i}(v) + \alpha_{i}(w)) x_{i}
\]

Since basis expansions are unique, this implies that \( \alpha_{i}(cv + w) = c \alpha_{i}(v) + \alpha_{i}(w) \) for all \( i \). Hence, \( \alpha_{i} \in V^{*} \). Since there are \( d \) such \( \alpha_{i}s \), showing that they constitute a basis requires only proving that they are linearly independent.

Suppose not—that is, suppose that there exists a set of coefficients \( \beta_{i} \in \mathbb{R} \) which are not all zero such that:

\[
0 = \sum_{i=1}^{d} \beta_{i} \alpha_{i}
\]

Letting \( j \) be chosen such that \( \beta_{j} \neq 0 \):

\[
\alpha_{j} = \sum_{i \neq j} \frac{\beta_{i}}{\beta_{j}} \alpha_{i}
\]

Since \( \alpha_{j}(x_{j}) = 1 \), and \( \alpha_{i}(x_{j}) = 0 \) for all \( i \neq j \), it follows that:

\[
1 = 0
\]

This is a contradiction. Hence, \( \alpha_{1}, \ldots, \alpha_{d} \) must be a basis for \( V^{*} \).

Another way of showing linear independence would be to consider an arbitrary \( f \in V^{*} \), and noting that by the linearity of \( f \), and taking the basis expansion of its argument:

\[
f(v) = f \left( \sum_{i=1}^{d} \alpha_{i}(v) x_{i} \right)
\]

\[
= \sum_{i=1}^{d} f(x_{i}) \alpha_{i}(v)
\]

Which shows that \( f \) may be represented as a linear combination of the elements \( \alpha_{i} \), with coefficients \( f(x_{i}) \). Since this is true for any \( f \in V^{*} \), it shows that \( \alpha_{1}, \ldots, \alpha_{d} \) spans \( V^{*} \), and therefore is a basis.
Uniqueness

Now suppose that $\beta_1, \ldots, \beta_d$ is a basis for $V^*$ such that $\beta_i(x_i) = 1$ and $\beta_i(x_j) = 0$ for all $i \neq j$.

Since $\alpha_i \in V^*$ and $\beta_1, \ldots, \beta_d$ are a basis for $V^*$, it follows that there exist coefficients $c_{i,1}, \ldots, c_{i,d}$ such that $\alpha_i = \sum_{j=1}^d c_{i,j} \beta_j$ for all $i$.

By the above, we have that $v = \sum_{i=1}^d \alpha_i(v) x_i$ for every $v \in V$. Hence, by the above:

$$\alpha_i(v) = \sum_{j=1}^d c_{i,j} \beta_j \left( \sum_{k=1}^d \alpha_k(v) x_k \right)$$

By the linearity of $\beta_j$:

$$\alpha_i(v) = \sum_{k=1}^d \alpha_k(v) \sum_{j=1}^d c_{i,j} \beta_j(x_k)$$

Since $\beta_j(x_k) = \delta_{jk}$, where $\delta$ is the Kronecker delta function:

$$\alpha_i(v) = \sum_{j=1}^d c_{i,j} \alpha_j(v)$$

Since this holds for all $v \in V$:

$$\alpha_i = \sum_{j=1}^d c_{i,j} \alpha_j$$

Finally, since $\alpha_1, \ldots, \alpha_d$ is a basis for $V^*$, the basis expansion of $\alpha_i$ must be unique, which implies that $c_{i,j} = \delta_{ij}$ for all $i, j$. By the definition of $c_{i,j}$:

$$\alpha_i = \sum_{j=1}^d c_{i,j} \beta_j$$

$$\alpha_i = \beta_i$$

Which shows that the basis $\alpha_1, \ldots, \alpha_d$ is unique.

Same holds in reverse

By the above, there is a unique basis $\hat{x}_1, \ldots, \hat{x}_d$ for $V^{**}$ which satisfies $\hat{x}_i(\alpha_i) = 1$ and $\hat{x}_i(\alpha_j) = 0$ if $i \neq j$. Mapping this basis under the natural isomorphism between $V^{**}$ and $V$ from the previous homework gives the desired result.

Problem 2

I will use the notation of the above solution to problem 1, rather than the notation in the problem set. For any $v \in V$ and $\beta \in V^*$, we may take the basis expansion of $v$:

$$\beta(v) = \beta \left( \sum_{i=1}^d \alpha_i(v) x_i \right)$$

By the linearity of $\beta$:

$$\beta(v) = \sum_{i=1}^d \alpha_i(v) \beta(x_i)$$
Since $\alpha_1, \ldots, \alpha_d$ is a basis for $V^*$, we may write $\beta = \sum_{i=1}^{d} c_i \alpha_i$:

$$\beta(v) = \sum_{i=1}^{d} \alpha_i(v) \sum_{j=1}^{d} c_j \alpha_j(x_i)$$

Since $\alpha_j(x_i) = \delta_{ij}$:

$$\beta(v) = \sum_{i=1}^{d} \alpha_i(v) c_i$$

This is the desired result.

**Problem 3**

For any $v, w \in V$, we may take the basis expansions of $v$ and $w$:

$$vMw = \left( \sum_{i=1}^{d} \alpha_i(v) x_i \right) M \left( \sum_{j=1}^{d} \alpha_j(w) x_j \right)$$

By the fact that $M$ is bilinear:

$$vMw = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i(v) \alpha_j(w) (x_i M x_j)$$

This is the desired result.

**Problem 4**

By problem 3, we may write:

$$vMw = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i(v) \alpha_j(w) (x_i M x_j)$$

For $x_1, \ldots, x_d$ a basis for $V$. We will prove existence of such a basis (though not a well-typed expression for it!), by orthogonalizing an existing basis. Accordingly, let $y_1, \ldots, y_d$ be a basis for $V$. I will now perform the Graham-Schmidt orthonormalization process on $y_1, \ldots, y_d$ to yield an orthonormal (relative to $M$) basis $x_1, \ldots, x_d$.

Define $z_1 = y_1$, and define $z_i = y_i - \sum_{j=1}^{i-1} \frac{y_i M z_j}{z_j M z_j} z_j$ for all $i > 1$. There are two things to note about this definition: first, that each $z_i$ is defined as a linear combination of $y_1, \ldots, y_i$, with the coefficient on $y_i$ being 1. Since the $y_i$s are linearly independent, this implies that the $z_i$s are linearly independent, and hence they constitute a basis. Also, note that $z_j M z_j > 0$ since $z_j \neq 0$, so the division in the definition of $z_i$ cannot cause any problems. To continue, I claim that $z_i M z_j = 0$ for all $j < i$. This will be a proof by induction. For $i = 1$, the claim holds trivially. Now suppose by the induction that $z_j M z_k = 0$ for all $k < j < i$, and consider the following for $k < i$:

$$z_i M z_k = \left( y_i M z_k - \sum_{j=1}^{i-1} \frac{y_i M z_j}{z_j M z_j} z_j M z_k \right)$$

By the induction, $z_j M z_k = 0$ if $j \neq k$ on the far RHS (inside the sum) above:

$$z_i M z_k = (y_i M z_k - y_i M z_k) = 0$$

This completes the induction, and proves that $z_1, \ldots, z_d$ are orthogonal relative to $M$. Finally, we define $x_i = \frac{z_i}{\sqrt{z_i M z_i}}$. Once again, since $z_i M z_i > 0$, the square-root, and the division, are perfectly safe. Since this is nothing
but a scaling of the orthogonal (relative to $M$) basis $z_i$, we must have that the $x$'s form an orthogonal (relative to $M$) basis, and furthermore that since $x_i M x_i = \frac{z_i M z_i}{(\sqrt{N})^2} = 1$, that $x_1, \ldots, x_d$ in fact form an orthnormal basis relative to $M$.

Now, for any $v, w \in V$, if we expand $v$ and $w$ in the basis $x_1, \ldots, x_d$:

$$v M w = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i(v) \alpha_j(w) (x_i M x_j)$$

Since $x_i M x_j = \delta_{ij}$:

$$v M w = \sum_{i=1}^{d} \alpha_i(v) \alpha_i(w)$$

This is the desired result. Note that unlike in the previous problems, we did not find a well-typed expression for this basis—we only proved its existence.

**Problem 5**

There are a huge number of possible answers to this problem. Here’s one. Consider the following:

$$f \overset{\Delta}{=} \lambda v : V . \text{the (w : V s.t. } W w = Q v)$$

Since $W$ is an inner product (in particular, it is of full-rank), it is invertible, showing that there is indeed a unique $w$ satisfying the above, for every $v$. Hence, the above expression is well-typed. To see that this is linear, we need only note that for all $u, v : V$ and $c : \mathbb{R}$:

$$W f (c v + u) = Q (c v + u) = c Q v + Q u = c W f (v) + W f (u)$$

And once more using the fact that $W$ is of full-rank, the above shows that $f (c v + u) = c f (v) = f (u)$. Since $W f = Q$, we see that we may interpret $f : V \rightarrow V$ as $W^{-1} Q$, where $W^{-1} : V^* \rightarrow V$ and $W, Q : V \rightarrow V^*$. 