1. Consider a graph structured factor graph with $V$ possible values at each node and where each node has $K$ neighbors. Give the order of run time of a single iteration of belief propagation as a function of $V$ and $N$. Explain your answer.

2. Consider a probabilistic context free grammar (PCFG) in Chomsky normal form so that every production has the form $X \rightarrow YZ$ or $X \rightarrow a$ where $X$, $Y$ and $Z$ are nonterminal symbols and $a$ is a terminal symbol. Suppose we assign a value $V(X \rightarrow XY)$ and $V(X \rightarrow a)$ to every production. Given a string $x$ of terminal symbols we want to find the parse tree of $x$ of largest value, i.e., such that the sum of the value of all productions in the parse tree is maximized.

   a. Consider a more traditional PCFG where each production has a probability $P(X \rightarrow YZ)$ such that for each nonterminal $X$ the sum of the probabilities of the productions from that nonterminal equals 1. Define the value function $V$ on productions such that the tree of maximum value is also the tree of largest probability.

   b. Given a value function $V$ on productions we define $V(y)$ for a parse tree $y$ to be the sum over the productions in $y$ of the value of that production. Define a feature map $\Phi(y)$ and a weight vector $w$ such that $V(y) = w \cdot \Phi(y)$. The dimension of $\Phi(y)$ should be independent of the length of the string generated by $y$.

3. Consider a linear dynamical system define by the following parameters:

   - $\gamma_1, P_1$ — the mean and covariance of the probability distribution of the initial state $y_1$.
   - $A, \Gamma$ — $A$ is a linear transformation on the state vectors and $\Gamma$ is a state vector covariance matrix. We have $y_{t+1} = Ay_t + \epsilon$ where $\epsilon$ is drawn from $\mathcal{N}(0, \Gamma)$.
   - $C, \Sigma$ — $C$ is a linear map from a state vector to an observation vector and $\Sigma$ is an observation vector covariance matrix. We have $x_t = Cy_t + \epsilon$ where $\epsilon$ is drawn from $\mathcal{N}(0, \Sigma)$.

   a. A linear dynamical system is stable if there exists a bound $b$ such that for arbitrarily large $t$ we have $\mathbb{E}[\|y_t\|^2] \leq b$. Give a condition on the matrix $A$ such that the system is stable.

   b. Now assume that the noise in the state transitions is isotropic, i.e., $\Gamma = \sigma^2 I$. Define a condition on $A$ such that there exists a stationary distribution for the internal state. In the case where a stationary distribution exists, solve for the stationary distribution as a function of $A$ and $\sigma^2$. Hint: The distribution is Gaussian and the eigenvectors of the covariance matrix of the stationary distribution are the same as the eigenvectors of $A$. 


