1. Given \((X, d_1)\) and \((Y, d_2)\) with \(X \cap Y = \{z\}\), the gate-sum of the two metrics is the metric \((X \cup Y, d)\), where \(d|_{X \times X} = d_1\), \(d|_{Y \times Y} = d_2\), and \(d(x, y) = d_1(x, z) + d_2(z, y)\) for \(x \in X\) and \(y \in Y\).

   (a) If both \(d_1\) and \(d_2\) are embeddable into \(\ell_1\), is \(d\) embeddable into \(\ell_1\)?
   (b) If both \(d_1\) and \(d_2\) are embeddable into \(\ell_2\), is \(d\) embeddable into \(\ell_2\)?

2. Show that Bourgain’s embedding also gives \((X, d) \xrightarrow{O(\log n)} \ell^p\). Use ideas from the proof for the \(\ell_2\)-embedding. Use Hölders inequality: \(\|x\|_p \cdot \|y\|_q \geq \langle x, y \rangle\), for \(1/p + 1/q = 1\).

Definition. A metric \((X, d)\) is called a squared \(\ell_2\)-metric if the metric \(d' = \sqrt{d}\) is an \(\ell_2\)-metric. In the following we denote the set of squared \(\ell_2\)-metrics with \(L_2^2\).

3. Show that any \(\ell_1\)-metric embeds into \(L_2\) isometrically.

4. Let \(H_\ell\) denote the hypercube of dimension \(\ell\), i.e., a graph with vertex set \(V(H_\ell) = \{0, 1\}^\ell\) and edge-set \(E(H_\ell) = \{(x, y) \in V \times V \mid x \text{ and } y \text{ differ in exactly one bit}\}\). Let \(d\) denote the shortest-path metric on \(H_\ell\). Show that \(d\) is an \(L_2\)-metric. How many dimension do you need for your embedding?

5. A metric \(d\) can be viewed as a function \(d : X \times X\) that assigns a length to each unordered pair of points in a space \(X\). We can encode this function by a point in \(\mathbb{R}^k\) with \(k = \binom{|X|}{2}\), i.e., any metric on \(n\) vertices corresponds to a point in \(\mathbb{R}^{\binom{n}{2}}\).

   (a) Show that the set of points in \(\mathbb{R}^k\) that correspond to a metric forms a convex cone.
   (b) What about points corresponding to \(\ell_1\)-metrics, \(\ell_2\)-metrics, \(\ell_\infty\)-metrics, \(L_2\)-metrics?

   For each case either prove that it forms a cone, or find a counter-example.

6. Prove that any \(n\)-point metric that is embeddable into \(\ell_1\) with any number of dimensions can actually be embedded into \(\ell_1^{\binom{n}{2}}\), i.e., using only \(\binom{n}{2}\) dimensions. Use the result for \(\ell_1\)-metrics from the previous question, the fact that \(\ell_1\)-metrics are cut-metrics, and apply some linear algebra.

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1 verify that \(d'\) is indeed a metric.

2 In the literature the set of squared \(\ell_2\)-metrics is often denoted with \(\ell_2^2\).