13 Improved cutting schemes for planar graphs

In this lecture, we will be studying randomized cutting procedures; given a graph $G = (V, E)$ and $\Delta > 0$, the (randomized) procedure returns a set of edges $E' \subseteq E$ such that

1. Every connected component of $G \setminus E'$ has a weak diameter of at most Δ.
2. For every $v \in V$, $\mathbf{Pr}[B(v, \rho)$ is cut by $E'] \leq \frac{1}{\beta} \cdot \frac{\rho}{\Delta}$.

As was shown in a previous lecture, $\beta = O(1/\log n)$ in general. Furthermore, it can be shown that $\beta = \Theta(1/\log n)$ is the best result possible. In this lecture, we will show that for a graphs excluding K_r as a minor, the parameter β can be improved to $\Omega(1/r^2)$. The first result in this vein was due to Klein, Plotkin and Rao (1995); they gave a cutting procedure and proved $\beta = \Omega(1/r^3)$ for it. The result was improved by Fakcharoenphol and Talwar (2003), who gave a more involved procedure guaranteeing $\beta = \Omega(r^2)$. In this lecture, we modify the analysis of Fakcharoenphol and Talwar, and show that the procedure of Klein, Plotkin and Rao actually gives $\beta = \Omega(1/r^2)$.

Note that by choosing $\rho = \Theta(\beta \cdot \delta)$ in the above definition of a cutting scheme we obtain $\Pr[B(v, \beta\delta)$ is cut$] \leq 1/2$ which is the definition of a cutting scheme used in the previous two lectures. The following corollary follows from the result of the previous lecture.

Corollary 13.1 *Any graph excluding K_r minors embeds into ℓ_1 with $O(r\sqrt{\log n})$ distortion.*

13.1 Preliminaries

Definition 13.2 *A graph H is a* minor *of graph G if H can be obtained from G by edge contractions and deletions.*

Graph minors have been studied extensively in graph theory; see, e.g., the book by Diestel (2000) for many results. The most famous result in graph minors is a proof (by Robertson and Seymour) of a conjecture of Wagner, which claims that any graph family closed under taking minors has a finite set of *excluded minors*. E.g., note that *planar* graphs (which are closed under taking minors) are exactly the set of graphs which exclude K_5 and $K_{3,3}$ as minors, and *outerplanar* graphs exclude K_4 and $K_{2,3}$ as minors.

Klein, Plotkin and Rao consider graphs which exclude $K_{r,r}$-minors, while Fakcharoenphol and Talwar consider graphs excluding K_r as minors. Note that a graph excluding K_r-minors also excludes $K_{r,r}$-minors, and a graph excluding $K_{r,r}$-minors also excludes K_{2r}-minors. Hence the two notions are equivalent up to a factor of 2; we will find it convenient to exclude K_r minors.

The proofs use an equivalent definition of graph minor for the analysis of the cutting procedure.

Definition 13.3 *Graph H is a minor of graph G if*

1. *For every $v \in V(H)$, there exists a connected subset of vertices $\mathcal{A}(v)$ in $V(G)$, which is known as a* super node*, such that for $v \neq w \in V(H)$, $\mathcal{A}(v)$ and $\mathcal{A}(w)$ are disjoint.*

2. *For every* $\{v, w\} \in E(H)$*, there exist* $v' \in \mathcal{A}(v)$ *and* $w' \in \mathcal{A}$ *such that* $\{v', w'\} \in E(G)$*, which is known as a* super edge*.*

13.2 The Cutting Procedure

For ease of notation, we let $\delta = \lceil \frac{\Delta}{r} \rceil$ and give a cutting procedure that given integers $r \geq 3$ and $\delta > 0$, and a graph $G = (V, E)$ excluding K_r-minors, returns $E' \subset E$ such that

1. Every connected component of $G \setminus E'$ has weak diameter $O(\delta r)$.
2. For every $v \in V$ and $\rho < O(\delta r)$, $Pr[B(v, \rho)$ is cut by $E'] \leq O(\frac{r\rho}{\delta})$.

Setting $\delta = \Delta / O(r^2)$ gives us the claimed decomposition. As usual, we make the simplifying assumption that every edge in G has unit length.

The KPR cutting procedure:

Given integers δ, r and graph $G = (V, E)$,
    Set $E' := \emptyset$;
    do repeat $r - 2$ times:
        Set $E'' := \emptyset$;
        for each connected component C of $G \setminus E'$:
            Perform BFS rooted at an arbitrary node a in C.
            Pick an integer k uniformly at random from $\{0, 1, \ldots, \delta - 1\}$.
            Let E_C be the set of edges at level $k \mod \delta$ from a.
            Set $E'' := E'' \cup E_C$;
        endfor
        Set $E' := E' \cup E''$;
    enddo
return E' as the set of deleted edges.

The first few iterations of the cutting procedure are shown in Figure 13.1. We denote the original graph by G_1. After the first iteration, deleting some edges causes the graph G_1 to be shattered into connected components. We use G_2 to denote one of these components. After a further iteration of the procedure, G_2 is shattered into connected components, one of which is denoted by G_3. Similarly, after the last iteration of the procedure, we have some connected component G_{r-1}. By $G_{r-1} \subset G_{r-2} \subset \cdots \subset G_2 \subset G_1$, we mean some hierarchy of subsets. (Note that there are many such hierarchies produced by the cutting procedure.) Moreover, we denote the root picked in G_i by a_i and the corresponding BFS tree by T_i.

13.3 The Analysis

Proposition 13.4 *For each* $v \in V$ *and* $\rho > 0$, $\mathbf{Pr}[B(v, \rho)$ *is cut by* $E'] \leq O(\frac{r\rho}{\delta})$*.*

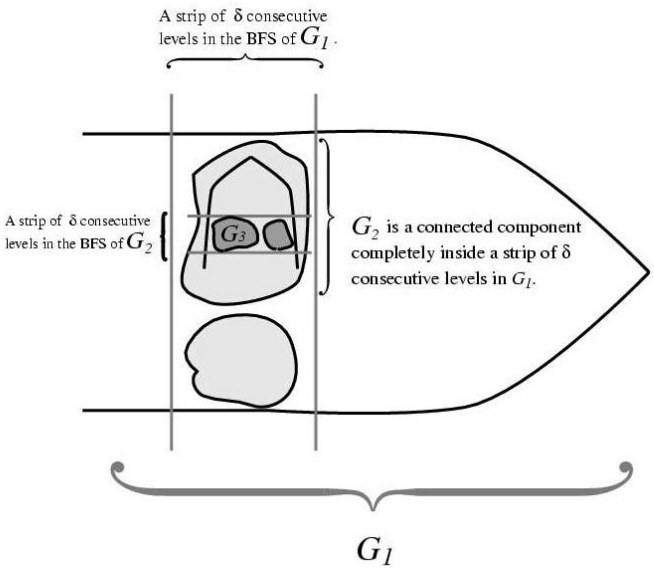

Figure 13.1: The first few iterations of the cutting procedure (taken from the paper of Klein et. al.)

Proof. Note that by triangle inequality,

$$\max\{d(a,u) : u \in B(v,\rho)\} - \min\{d(a,u) : u \in B(v,\rho)\} \leq 2\rho.$$

Hence, at each level of recursion, $\mathbf{Pr}[B(v,\rho)$ is cut by $E''] \leq O(\frac{\rho}{\delta})$. The result follows by an application of the trivial union bound. ■

Hence, it remains to show that each connected component produced by the cutting procedure has small diameter. We first prove a few lemmas.

Lemma 13.5 (Moat Argument) *Suppose $u, v \in G_{i+1}$ and u is an ancestor of v in tree T_i. Then, u and v are within δ levels of each other in tree T_{i+1}. Equivalently, if u and v are more than δ levels apart in T_{i+1}, then one cannot be an ancestor of the other in T_i.*

Proof. Since $u, v \in G_{i+1}$, it follows that u and v must be within δ levels of each other in G_i. Moreover, u is an ancestor of v in T_i, and so there exists a path of length less than δ from u to v in G_{i+1}. Hence, in any BFS tree in G_{i+1}, u and v must be within δ levels of each other. ■

Lemma 13.6 *Suppose G_{r-1} contains two nodes a_{r-1} and a_r such that $d_G(a_{r-1}, a_r) > (8r+2)\delta$. Then, for $1 \leq i < j \leq r$, we have $d_G(a_i, a_j) > 4\delta r$.*

Proof. For $i = r-1$, the result is trivial. Consider $1 \leq i \leq r-2$. Suppose $i < j$ and $d_G(a_i, a_j) \leq 4\delta r$. Since a_j, a_{r-1} and a_r are inside the connected component G_{i+1}, it follows that they are within δ levels of one another in T_i. Hence, $d_G(a_i, a_j) \leq 4\delta r$ implies that $d_G(a_{r-1}, a_r) \leq (8r+2)\delta$. ■

Lemma 13.7 *Suppose G_{r-1} contains two nodes a_{r-1} and a_r such that $d_G(a_{r-1}, a_r) > (8r+2)\delta$. Then, for $b = r-2, r-3, \ldots, 2, 1$, the following statements hold.*

1. *There is a K_{r-b}-minor in G_{b+1} such that for each $b+1 \leq j \leq r$, there exists a super node $\mathcal{A}(a_j)$ containing a_j in G_{b+1}.*
2. *For each $b+1 \leq j \leq r$, there exists a path P_j of length 4δ starting at some node in $\mathcal{A}(a_j)$ and going towards a_b in T_b, such that the paths are pairwise disjoint.*
3. *Each path P_j is disjoint from $\mathcal{A}(a_k)$ for $j \neq k$.*
4. *For $j \neq k$, the middle nodes h_j and h_k of the respective paths P_j and P_k are more than $4b\delta$ apart in G.*
5. *For each j, $d_G(h_j, a_b) > 4b\delta$.*

Proof. We prove the result by (reverse) induction on b. Note that we assume $r \geq 3$.

For $b = r-2$, consider a shortest path P from a_{r-1} to a_r in the connected component G_{r-1}.

1. By assumption, the length of P is greater than $(8r+2)\delta$. The super nodes $\mathcal{A}(a_{r-1})$ and $\mathcal{A}(a_r)$ can be formed by splitting the path P halfway, with nodes nearer to a_{r-1} in $\mathcal{A}(a_{r-1})$ and nodes nearer to a_r in $\mathcal{A}(a_r)$.
2. For $j \in \{r-1, r\}$, define path P_j to be the path of length 4δ starting from a_j and going towards a_{r-2} in T_{r-2}. Since $d_G(a_{r-1}, a_r) > (8r+2)\delta > 8\delta$, the paths P_{r-1} and P_r must be disjoint.
3. Note that for $j \neq k$, the path P_j starts from a_j and leaves G_{r-1} within δ steps. Since the shortest path in G_{r-1} from a_j to $\mathcal{A}(a_k)$ is at least $4\delta r$, P_j is disjoint from $\mathcal{A}(a_k)$.
4. Since $d_G(a_{r-1}, a_r) > (8r+2)\delta$, the middle nodes h_{r-1} and h_r must be more than $(8r-2)\delta > 4(r-2)$ apart, by the triangle inequality.
5. By Lemma 13.6, for $j \in \{r-1, r\}$, we have $d_G(a_{r-1}, a_j) > 4\delta r$ and so by the triangle inequality, we have $d_G(h_j, a_{r-1}) > 4\delta r - 2\delta > 4(r-2)\delta$.

For the inductive step, assume the result holds for $b = i+1$, where $i \geq 1$. We prove the result holds for $b = i$. fFigure 13.2 shows the case for $i = r-4$.

1. For $j \in \{i+2, \ldots, r\}$, extend $\mathcal{A}(a_j)$ to include all but the last node on P_j to form $\mathcal{A}'(a_j)$. By statements 2 and 3 of the induction hypothesis, these extended super nodes are pairwise disjoint. The new super node $\mathcal{A}'(a_{i+1})$ is formed by including the vertices in the path in T_{i+1} from a_{i+1} down to the last node in P_j for each $j \in \{i+2, \ldots, r\}$. Note that $\mathcal{A}'(a_{i+1})$ is disjoint from G_{i+2} and so by construction is disjoint from other extended super nodes $\mathcal{A}'(a_j)$. The last edge in P_j is the super edge connecting $\mathcal{A}'(a_j)$ and $\mathcal{A}'(a_{i+1})$. Hence, the super nodes $\{\mathcal{A}'(a_j) : j = i+1, \ldots, r\}$ form a K_{r-i} minor in G_{i+1}.

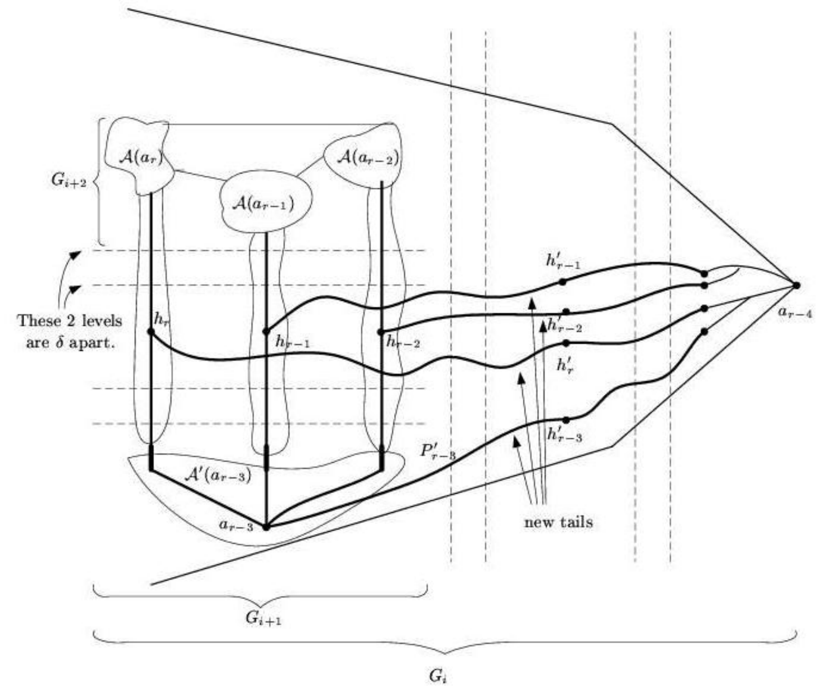

Figure 13.2: The Inductive Step (taken from the paper of Fakcharoenphol and Talwar)

2. For $j \in \{i+2, \ldots, r\}$, define P'_j to be the path of length 4δ starting from h_j and going towards a_i in T_i. By statement 4 of the induction hypothesis, these paths must be disjoint. Define P'_{i+1} to be the path of length 4δ starting from a_{i+1} and going towards a_i in T_i. By statement 5 of the induction hypothesis, P'_{i+1} must be disjoint from the other paths P'_js.

3. First consider $j \neq k \in \{i+2, \ldots, r\}$. By statement 2 of the induction hypothesis, P'_j must be disjoint from P_k. Furthermore, h_j is more than δ levels away from any node in G_{i+2} in T_{i+1}. Hence, by the Moat Argument, any node in G_{i+2} cannot be an ancestor of h_j in T_i. Therefore, P'_j is disjoint from $\mathcal{A}(a_k)$. So, P'_j is disjoint from $\mathcal{A}'(a_k)$. Similarly, h_j is more than δ levels away from any node in $\mathcal{A}'(a_{i+1})$. Hence, using the Moat Argument again, we can show P'_j is disjoint from $\mathcal{A}'(a_{i+1})$.

   By statement 5 of the induction hypothesis, P'_{i+1} cannot intersect P_j. Moreover, a_{i+1} is more than δ levels from any node in G_{i+2}. Hence, by the Moat Argument again, P'_{i+1} is disjoint from $\mathcal{A}(a_j)$. Therefore, P'_{i+1} is disjoint from $\mathcal{A}'(a_j)$.

4. First consider $j \neq k \in \{i+2, \ldots, r\}$. Note that both $d_G(h_j, h'_j)$ and $d_G(h_k, h'_k)$ are at most 2δ. Hence, by statement 4 of the induction hypothesis and the triangle inequality, $d_G(h'_j, h'_k) \geq d_G(h_j, h_k) - 4\delta > 4i\delta$. Similarly, by statement 5 of the induction hypothesis, we have $d_G(h'_j, h'_{i+1}) > 4i\delta$.

5. Note that for each $j \in \{i+1, i+2, \ldots, r\}$, we have $d_G(a_j, h'_j) \leq 2\delta(r-1-i)$, since at each

iteration a middle point moves at most 2δ. By Lemma 13.6, $d_G(a_j, a_i) > 4\delta r$. Hence, by the triangle inequality, $d_G(a_i, h'_j) > 4\delta r - 2\delta(r-1-i) > 4i\delta$.

This completes the inductive step. ■

Proposition 13.8 *Suppose G_{r-1} has weak diameter larger than $(8r+2)\delta$. Then, graph G contains a K_r-minor.*

Proof. The assumption implies the condition of Lemma 13.7. In particular, the conclusion of Lemma 13.7 holds for $b = 1$. Extending the super nodes $\mathcal{A}(a_j)$ for $j \in \{2, 3, \ldots, r\}$ and defining a new super node $\mathcal{A}'(a_1)$ in the same way as in the proof of statement 1 for the inductive step in Lemma 13.7 give the required K_r-minor. ■

Propositions 13.4 and 13.8 together give the main result asserted at the beginning of this section.

References

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