

Algorithmic regularity for polynomials and applications

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Abstract

In analogy with the regularity lemma of Szemerédi [Sze75], regularity lemmas for polynomials shown by Green and Tao [GT09] and by Kaufman and Lovett [KL08] give a way of modifying a given collection of polynomials $\mathcal{F} = \{P_1, \dots, P_m\}$ to a new collection \mathcal{F}' so that the polynomials in \mathcal{F}' are “pseudorandom”. These lemmas have various applications, such as (special cases) of Reed-Muller testing and worst-case to average-case reductions for polynomials. However, the transformation from \mathcal{F} to \mathcal{F}' is not algorithmic for either regularity lemma. We define new notions of regularity for polynomials, which are analogous to the above, but which allow for an efficient algorithm to compute the pseudorandom collection \mathcal{F}' . In particular, when the field is of high characteristic, in polynomial time, we can refine \mathcal{F} into \mathcal{F}' where every nonzero linear combination of polynomials in \mathcal{F}' has desirably small Gowers norm.

Using the algorithmic regularity lemmas, we show that if a polynomial P of degree d is within (normalized) Hamming distance $1 - \frac{1}{|\mathbb{F}|} - \varepsilon$ of some unknown polynomial of degree k over a prime field \mathbb{F} (for $k < d < |\mathbb{F}|$), then there is an efficient algorithm for finding a degree- k polynomial Q , which is within distance $1 - \frac{1}{|\mathbb{F}|} - \eta$ of P , for some η depending on ε . This can be thought of as decoding the Reed-Muller code of order k *beyond* the list decoding radius, in the sense of finding one close codeword, when the received word P itself is a polynomial (of degree larger than k but smaller than $|\mathbb{F}|$).

We also obtain an algorithmic version of the worst-case to average-case reductions by Kaufman and Lovett [KL08]. They show that if a polynomial of degree d can be weakly approximated by a polynomial of lower degree, then it can be computed exactly using a collection of polynomials of degree at most $d - 1$. We give an efficient (randomized) algorithm to find this collection.

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1 Introduction

Regularity is a notion of “pseudorandomness” that allows one to decompose a given object into a collection of simpler objects which appear random according to certain statistics. The famous regularity lemma of Szemerédi [Sze75, Sze78] says that any dense graph can be partitioned into a collection of bounded number of “pseudorandom” bipartite graphs. The Szemerédi regularity lemma has numerous applications in combinatorics and property testing.

The original proof by Szemerédi was non-algorithmic and the question of finding an algorithm for computing the regularity partition was first considered by Alon et al. [ADL⁺94], motivated (at least partly) by the problem of converting some of the applications of the regularity lemma into algorithms. The regularity lemma is often used to guarantee the existence of certain structures in a graph, and an algorithmic version of the lemma allows one to *find* these structures in a given graph. Since then, there have been numerous improvements and extensions to the algorithmic version of the Szemerédi regularity lemma, of which the works [AN06, FK99, KRT03, FMS10] constitute a partial list (see [FMS10] for a detailed discussion).

In studying a special case of the inverse conjecture for Gowers’ norms over finite fields, Green and Tao [GT09] introduced a notion of regularity for a collection of polynomials. We call a collection $\mathcal{F} = \{P_1, \dots, P_m\}$ of a bounded number of polynomials over \mathbb{F}^n for a prime field \mathbb{F} , a **factor** of degree d , if all the polynomials in the collection have degree at most d . The Green-Tao notion of regularity states that a given factor is regular if every linear combination $\sum_{i=1}^m c_i \cdot P_i$ of the polynomials in the given factor has high rank i.e., if $k \leq d$ is the degree of the highest-degree polynomial with a non-zero coefficient in the linear combination $Q = \sum_i c_i \cdot P_i$, then Q cannot be written as a function Γ of some bounded number M of degree- $(k-1)$ polynomials. Their notion in fact allows M to be a *function of the number of polynomials in \mathcal{F}* (which is m here).

Green and Tao show that any given factor \mathcal{F} of bounded degree- d with the number of polynomials m denoted as $\dim(\mathcal{F})$, and a function $F : \mathbb{N} \rightarrow \mathbb{N}$, one can “refine” it to a factor \mathcal{F}' of degree- d which is F -regular i.e., it is regular in the above sense, with the lower bound on the rank given by $F(\dim(\mathcal{F}'))$. Here by \mathcal{F}' being a refinement of \mathcal{F} , we mean that each polynomial P in \mathcal{F} can be expressed as a function of polynomials from \mathcal{F}' . Also, they show that the number of polynomials in \mathcal{F}' depends only on $d, \dim(\mathcal{F})$ and the function F , but is independent of the number of variables n . This can again be compared with the Szemerédi regularity lemma, which is proved by showing that starting from any partition of a given graph, one can “refine” it to a regular partition, such that the number of pieces only depend on the regularity parameter and the number of pieces in the original partition.

We consider (and solve) the question of finding an algorithmic version of the above regularity lemma, motivated by one of the applications in [GT09]. For $c \in \mathbb{F}$, let $e_{\mathbb{F}}(c)$ denote $e^{2\pi ic/|\mathbb{F}|}$, where i is the square root of -1 . For $f : \mathbb{F}^n \rightarrow \mathbb{C}$, let $\Delta_h(f)(x) = f(x+h)\overline{f(x)}$ denote the multiplicative *derivative* of f in the direction h . Then $\|f\|_{U^{k+1}}$, the $(k+1)$ th Gowers norm of f , is defined as

$$\|f\|_{U^{k+1}} \stackrel{\text{def}}{=} \left| \mathbb{E}_{x, h_1, \dots, h_k \in \mathbb{F}^n} [\Delta_{h_1} \dots \Delta_{h_k} f(x)] \right|^{1/2^k}.$$

Green and Tao show the following:

Theorem 1.1 ([GT09]) *If a polynomial P of degree $d < |\mathbb{F}|$ satisfies $\|e_{\mathbb{F}}(P)\|_{U^{k+1}} \geq \varepsilon$, then there exists a polynomial Q of degree at most k , such that*

$$|\langle e_{\mathbb{F}}(P), e_{\mathbb{F}}(Q) \rangle| = \left| \mathbb{E}_{x \in \mathbb{F}^n} [e_{\mathbb{F}}(P(x) - Q(x))] \right| \geq \eta,$$

for some η depending only on ε and d .

Testing and Decoding Reed-Muller codes beyond the list-decoding radius. The above result has a direct interpretation in terms of Reed-Muller codes over \mathbb{F} , which raises the algorithmic question we consider. The Reed-Muller code of order k over \mathbb{F}^n is simply the set of polynomials of degree at most k over \mathbb{F}^n . If for a given polynomial P of degree d , there exists a polynomial Q of degree k such that $\mathbb{P}_{x \in \mathbb{F}^n} [P(x) = Q(x)] \geq \frac{1}{|\mathbb{F}|} + \varepsilon$, which is the same as saying that $\text{Dist}(P, Q) \leq 1 - \frac{1}{|\mathbb{F}|} - \varepsilon$ (for $\text{Dist}(P, Q)$ denoting the normalized Hamming distance), then it follows from the definition of Gowers norms and the Gowers Cauchy-Schwarz inequality that $\|e_{\mathbb{F}}(tP)\|_{U^{k+1}} \geq \varepsilon$ for a nonzero $t \in \mathbb{F}$. Then, [Theorem 1.1](#) gives that there exists a polynomial \tilde{Q} of degree k such that $\left| \langle e_{\mathbb{F}}(tP), e_{\mathbb{F}}(\tilde{Q}) \rangle \right| \geq \eta$, which can be translated to saying that $\text{Dist}(P, Q') \leq 1 - \frac{1}{|\mathbb{F}|} - \eta'$ for some $\eta' > 0$ and Q' of degree k .

Thus, the Gowers norm gives an approximate test for checking if for a given P , there exists a Q of degree at most k within Hamming distance $1 - \frac{1}{|\mathbb{F}|} - \varepsilon$. If there exists a Q , then the Gowers norm is large and if the Gowers norm is larger than ε , then there exists a Q' within distance $1 - \frac{1}{|\mathbb{F}|} - \eta'$. This is remarkable because the list-decoding radius of Reed-Muller of order k codes is only $1 - \frac{k}{|\mathbb{F}|}$ for $k < |\mathbb{F}|$ [[Gop10](#)], and the test works even beyond that. In fact the Hamming distance of a random P is $1 - \frac{1}{|\mathbb{F}|} - o(1)$ from all Q of degree k and the test works all the way up to that distance. However, note that [Theorem 1.1](#) only shows that this test works for a P which is a polynomial of degree $d < |\mathbb{F}|$. Tao and Ziegler [[TZ10](#)] later showed that this test works even when P is an arbitrary function and $k < |\mathbb{F}|$. For fields of low characteristic, the only general testing results for all functions of the above flavor (which works beyond the list-decoding radius) were proved by Samorodnitsky [[Sam07](#)] for Reed-Muller codes of order 2 over \mathbb{F}_2^n and by Green and Tao [[GT08](#)] for Reed-Muller codes of order 2 over \mathbb{F}_5^n .

Given the above, it is natural to consider the *decoding* analogue of the above question:

Given P of degree d over \mathbb{F}^n , if there exists Q of degree k such that $\text{Dist}(P, Q) \leq 1 - \frac{1}{|\mathbb{F}|} - \varepsilon$, can one find a Q' (in time polynomial in n) such that $\text{Dist}(P, Q') \leq 1 - \frac{1}{|\mathbb{F}|} - \eta$ for some η depending on ε ?

Note that d, k and $|\mathbb{F}|$ are assumed to be constants and dependence on these is allowed, but not on n . Also, observe that there might be exponentially many such Q since we are in the regime beyond the list-decoding radius (see [[KLP12](#)]), but the question turns out to be tractable since we only ask for one such Q and allow a loss from ε to η .

Such a decoding question was solved for Reed-Muller codes of order 2 over \mathbb{F}_2 , for any given function f (instead of a polynomial P of bounded degree) by [[TW11](#)]. We solve the above decoding question for polynomials P of degree d and the Reed-Muller code of order k for $k \leq d < |\mathbb{F}|$. This special case can be interpreted as follows: if P is of degree- d for some degree $k \leq d < |\mathbb{F}|$, then we can think of P being obtained from some Q of degree k (which is a codeword) by adding the “noise” $P - Q$ of degree d . Thus, when the noise is “structured” i.e., given by a degree- d polynomial, we can decode in the above sense (of finding *some* codeword within a given distance) even beyond the list-decoding radius.

Both our algorithmic version of the regularity lemma and the decoding algorithm, are randomized algorithms that run in time $O(n^d)$ and output the desired objects with high probability. This is linear time for regularity since even writing a polynomial of degree d takes time $\Omega(n^d)$. For the

question of finding a degree k polynomial within a given distance of P , it is possible that one may be able to do this in time $O(n^k)$, but we do not achieve this.

We remark that we do not give a regularity lemma for the notion of regularity stated by Green and Tao. We in fact define a related analytical notion of regularity, and show that one can prove an efficient regularity lemma for this notion, and also that this notion suffices for their application and our algorithmic version of it. However, our algorithmic version of the regularity lemma (for this notion) only works when $|\mathbb{F}| > d$. This is because our notion of regularity is based on the Gowers norm, and even to prove the regularity lemma, one needs to use (and modify) the proof of Green and Tao for the inverse theorem in [GT09], and their proof only works when $|\mathbb{F}| > d$.

We define other notions later, which work over small fields. However, the above notion based on Gowers norms is conceptually much simpler and may be useful for other applications. We would like to point out that a similar notion of regularity, called analytic rank, was introduced by Gowers and Wolf [GW10] and later used by Tao and Ziegler [TZ12] in their proof of inverse theorem for Gowers norms.

Regularity for Low Characteristics and Efficient Worst-Case to Average-Case Reductions for Polynomials. Kaufman and Lovett [KL08] develop a more involved notion of regularity to carry over part of the Green-Tao result to the setting when $|\mathbb{F}|$ is small.

The technical part of the Green-Tao proof proves the following result: If P is a polynomial of degree d such that $\text{bias}(P) = |\mathbb{E}_x [e_{\mathbb{F}}(P(x))]| \geq \delta$, then there exist polynomials Q_1, \dots, Q_M of degree at most $d - 1$, and a function $\Gamma : \mathbb{F}^M \rightarrow \mathbb{F}$ such that $P = \Gamma(Q_1, \dots, Q_M)$. Here M depends only on d, δ and $|\mathbb{F}|$. This can be read as saying that a biased degree- d polynomial can be computed by few polynomials of degree $d - 1$.

Kaufman and Lovett [KL08] manage to derive the above conclusion even when $|\mathbb{F}|$ is small, by defining and analyzing a more sophisticated notion of regularity. The notion is bit technical and we defer the description to [Section 3](#). They also use their analog of the above result to derive an interesting application. Let P be a degree- d polynomial which can be *weakly approximated* by a lower degree polynomial Q , of degree $k < d$. Here by weak approximation we mean $\langle e_{\mathbb{F}}(P), e_{\mathbb{F}}(Q) \rangle \geq \delta$. Then Kaufman and Lovett show that P can in fact be *computed* by lower degree polynomials i.e., there exist Q_1, \dots, Q_m of degree at most $d - 1$ and a function $\Gamma : \mathbb{F}^M \rightarrow \mathbb{F}$, such that $P = \Gamma(Q_1, \dots, Q_M)$. Again, M depends only on d, δ and $|\mathbb{F}|$.

We consider the question of whether it is possible to *find* the polynomials Q_1, \dots, Q_M and the function Γ efficiently? To this end, we define a version of the Kaufman-Lovett regularity notion, for which we can prove an algorithmic regularity lemma over low characteristics. We also use this lemma to give an efficient algorithm for finding the above polynomials Q_1, \dots, Q_M with high probability.

We believe that our notions of regularity and the algorithmic regularity lemmas for these will also be useful in other applications. We collect the various notions of regularity and provide a more technical overview in [Section 3](#). We also provide a brief overview of the proofs below.

Proof Overview

The key technical part of both the Green-Tao result and the Kaufman-Lovett result is proving that a biased polynomial of degree d can be computed by a few polynomials of degree at most $d - 1$. Both proofs proceed by applying a lemma of Bogdanov and Viola [BV07] to find a factor \mathcal{F} of

degree $d - 1$ such that the polynomials in \mathcal{F} compute P correctly on *most* inputs $x \in \mathbb{F}^n$. Then both proofs proceed to refine \mathcal{F} to \mathcal{F}' by their respective notions of regularity, and show that the polynomials in \mathcal{F}' must in fact compute P exactly. It is an easy observation, which we make in [Section 2](#), that the Bogdanov-Viola lemma is algorithmic.

What remains is then to show that one can refine \mathcal{F} to \mathcal{F}' efficiently. As discussed before this is not possible using the original notions of regularity, and we achieve this by defining related but somewhat different notions of regularity. For these notions, we are able to go from \mathcal{F} to \mathcal{F}' efficiently. But then, we also need to prove that \mathcal{F}' which is now regular according to our new notion of regularity, still computes P exactly. For this we need to show that the Green-Tao (resp. Kaufman-Lovett) proof goes through with our notion. This requires a tighter analysis of both the proofs by Green-Tao and Kaufman-Lovett, and amounts to showing that the new notions of regularity, although are weaker, they still obey similar equidistribution properties that are used in both proofs.

The key difficulty in proving an algorithmic regularity lemma for polynomials is the same as that in the case of Szemerédi's lemma for graphs: proving a certificate of non-regularity. For example, in the Green-Tao notion, a factor $\mathcal{F} = \{P_1, \dots, P_m\}$ is *not regular*, if a linear combination $\sum_i c_i \cdot P_i$ can be expressed as function $\Gamma(Q_1, \dots, Q_M)$ for some lower degree polynomials Q_1, \dots, Q_M and an appropriate M . They then proceed by adding Q_1, \dots, Q_M to the factor and removing one of the P_i s. However, to provide an algorithmic version, one needs to *find* Q_1, \dots, Q_M , which is not an easy problem.

We get around this by defining a notion of regularity which requires the Gowers norm of every linear combination to be small (we call notion γ -uniformity). Now it is easy to check if some linear combination (of say degree d) has large Gowers norm. However, if this happens, it is not clear how to proceed to refine \mathcal{F} . We then look at the proof of Green and Tao, who show that when a polynomial has large Gowers norm, then an appropriate derivative of it (when we also think of the direction for the derivative as a variable) has large bias. This means that this derivative can now be computed on most inputs by polynomials of degree $d - 1$ (using Bogdanov-Viola), forming a factor \mathcal{F}_{d-1} of degree $d - 1$. *By induction, we can assume that we can in fact refine \mathcal{F}_{d-1} to a regular \mathcal{F}'_{d-1} factor of degree $d - 1$.* If our notion of regularity is good enough to show that a regular \mathcal{F}_{d-1} must exactly compute the derivative (and hence also the linear combination), we can add all the polynomials in \mathcal{F}_{d-1} to \mathcal{F} and proceed. Note that because of the nature of this induction, the proof of the Green-Tao result and the refinement of a factor to a regular one, are linked and thus we only obtain an algorithmic lemma with this notion when $|\mathbb{F}| > d$.

For the Kaufman-Lovett version, our notion simply says that the bias of certain linear combinations of polynomials in the factor and their derivatives is small,

along with a few other technical conditions imposed in [\[KL08\]](#). Although the notion of regularity here is more complicated, the proof of the algorithmic regularity lemma is in fact simpler and does not involve an induction as in the above case. However, showing that our notion suffices for their application requires some work. We defer the details to [Section 3](#) and [Section 6](#).

1.1 Definitions

Definition 1.2 (Factors) *Let $d \geq 0$ and M_1, \dots, M_d be non-negative integers. By factor of degree d on \mathbb{F}^n we mean a collection $\mathcal{F} = (P_{i,j})_{1 \leq i \leq d, 1 \leq j \leq M_i}$ of polynomials where $\deg(P_{i,j}) = i$ for all i, j . By the dimension vector of \mathcal{F} we mean $(M_1, \dots, M_d) \in \mathbb{N}^d$ and by the dimension of \mathcal{F} denoted by $\dim(\mathcal{F})$ we mean the number of polynomials in the factor, namely $M_1 + \dots + M_d$.*

Every factor \mathcal{F} defines a σ -algebra $\sigma(\mathcal{F})$ defined by atoms of the form $\{x : P_{i,j}(x) = c_{i,j}\}$. We write $\Sigma = \mathbb{F}^{M_1} \times \dots \times \mathbb{F}^{M_d}$ and call this the configuration space of \mathcal{F} . Moreover we let $\|\mathcal{F}\| \stackrel{\text{def}}{=} |\Sigma|$. By abuse of notation we write $\mathcal{F} : \mathbb{F}^n \rightarrow \Sigma$ for the evaluation map $\mathcal{F}(x) = (P_{i,j}(x))_{1 \leq i \leq d, 1 \leq j \leq M_i}$.

Sometimes if we do not make use of the dimension vector of \mathcal{F} explicitly, we may write $\mathcal{F} = \{P_1, \dots, P_m\}$. In this notation, we will simply use i as index of the polynomial P_i , and not to denote the degree of P_i .

Definition 1.3 (Measurability) Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a factor of degree d and let $f : \mathbb{F}^n \rightarrow \mathbb{F}$ be any given function. We say that f is measurable in \mathcal{F} if there exists a function $\Gamma : \mathbb{F}^m \rightarrow \mathbb{F}$ such that $f = \Gamma(P_1, \dots, P_m)$.

We say that f is σ -close to being measurable if there exists a Γ such that $\mathbb{P}_{x \in \mathbb{F}^n} [f(x) \neq \Gamma(P_1(x), \dots, P_m(x))] \leq \sigma$.

Definition 1.4 (Refinement) Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a factor of degree d . We say that a factor $\mathcal{F}' = \{Q_1, \dots, Q_M\}$ of degree- d is a refinement of \mathcal{F} if each polynomial P_i in \mathcal{F} is measurable in \mathcal{F}' i.e., there exists $\Gamma : \mathbb{F}^M \rightarrow \mathbb{F}^m$, so that $(P_1, \dots, P_m) = \Gamma(Q_1, \dots, Q_M)$.

We say that \mathcal{F}' is σ -close to being a refinement of \mathcal{F} , if there exists $\Gamma : \mathbb{F}^M \rightarrow \mathbb{F}^m$, so that $\mathbb{P}_{x \in \mathbb{F}^n} [(P_1(x), \dots, P_m(x)) \neq \Gamma(Q_1(x), \dots, Q_M(x))] \leq \sigma$.

Definition 1.5 (Derivatives) For a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ and a point $h \in \mathbb{F}^n$, we define the derivative of P in direction h as the function $D_h P(x) = P(x+h) - P(x)$. Note that if P is of degree d , then $D_h P$ is of degree $d-1$.

For a function $f : \mathbb{F}^n \rightarrow \mathbb{C}$, we will use the derivative to mean the multiplicative derivative defined as $\Delta_h f(x) = f(x+h)f(x)$. Note that if $f = e_{\mathbb{F}}(P)$ for some polynomial P , then $\Delta_h f = e_{\mathbb{F}}(D_h P)$.

2 Approximating a biased polynomial

The Bogdanov-Viola Lemma [BV07] states that if a polynomial of degree d is biased then it can be approximated by a bounded set of polynomials of lower degree. The following is an easy to observe algorithmic version of this lemma. We follow the proof of the lemma by Green and Tao [GT09].

Lemma 2.1 (Algorithmic Bogdanov-Viola lemma) Let $d \geq 0$ be an integer, and $\delta, \sigma, \beta \in (0, 1]$ be parameters. There exists a randomized algorithm, that given query access to a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree d with

$$\text{bias}(P) \geq \delta,$$

runs in time $O_{\delta, \beta, \sigma}(n^d)$, and with probability $1 - \beta$ returns functions $\tilde{P} : \mathbb{F}^n \rightarrow \mathbb{F}$ and $\Gamma : \mathbb{F}^C \rightarrow \mathbb{F}$, and a set of polynomials P_1, \dots, P_C , where $C \leq \frac{|\mathbb{F}|^5}{\delta^2 \sigma \beta}$ and $\deg(P_i) < d$ for all $i \in [C]$, for which

- $\mathbb{P}_x(P(x) \neq \tilde{P}(x)) \leq \sigma$, and
- $\tilde{P}(x) = \Gamma(P_1(x), \dots, P_C(x))$.

Proof: The proof will be an adaptation of the proof from [GT09]. Given query access to the polynomial P , we can compute the explicit description of P in $O(n^d)$ queries. For every $a \in \mathbb{F}$

define the measure $\mu_a(t) \stackrel{\text{def}}{=} \mathbb{P}(P(x) = a + t)$. It is easy to see that if $\text{bias}(P(x)) \geq \delta$ then, for every $a \neq b$,

$$\|\mu_a - \mu_b\| \geq \frac{4\delta}{|\mathbb{F}|}. \quad (1)$$

We will try to estimate each of these distributions. Let

$$\tilde{\mu}_a(t) \stackrel{\text{def}}{=} \frac{1}{C} \sum_{1 \leq i \leq C} \mathbb{1}_{P(x_i)=a+t},$$

where $C > \frac{|\mathbb{F}|^5}{\delta \beta_1}$, and $x_1, x_2, \dots, x_C \in \mathbb{F}^n$ are chosen uniformly at random. Therefore by an application of Chebyshev's inequality

$$\mathbb{P} \left(|\tilde{\mu}_a(t) - \mu_a(t)| > \frac{\delta}{2|\mathbb{F}|^2} \right) < \frac{\beta_1}{|\mathbb{F}|},$$

for all $t \in \mathbb{F}$ and therefore

$$\mathbb{P} \left(\|\tilde{\mu}_a - \mu_a\| > \frac{\delta}{2|\mathbb{F}|^2} \right) < \beta_1. \quad (2)$$

Now we will focus on approximating $P(x)$. Remember that $D_h P(x) = P(x+h) - P(x)$ is the additive derivative of $P(x)$ in direction h . We have

$$\mathbb{P}_h(D_h P(x) = r) = \mathbb{P}_h(P(x+h) - P(x) = r) = \mu_{P(x)}(r),$$

where $h \in \mathbb{F}^n$ is chosen uniformly at random. Let $\mathbf{h} = (h_1, \dots, h_C) \in (\mathbb{F}^n)^C$ be chosen uniformly at random, where C is a sufficiently large constant to be chosen later. Define

$$\mu_{\text{obs}}^{(x)}(t) \stackrel{\text{def}}{=} \frac{1}{C} \sum_{1 \leq i \leq C} \mathbb{1}_{D_{h_i} P(x)=t},$$

and let

$$\tilde{P}_h(x) \stackrel{\text{def}}{=} \arg \min_{r \in \mathbb{F}} \left\| \tilde{\mu}_r - \mu_{\text{obs}}^{(x)} \right\|.$$

Now choosing $C \geq \frac{|\mathbb{F}|^5}{\delta^2 \sigma \beta_2}$ follows

$$\mathbb{P}_h(\tilde{P}_h(x) \neq P(x)) \leq \mathbb{P}_h \left(\left\| \mu_{\text{obs}}^{(x)} - \mu_{P(x)} \right\| \geq \frac{\delta}{|\mathbb{F}|} \right) \leq \sigma \beta_2, \quad (3)$$

where the first inequality follows from (1) and (2). Therefore

$$\mathbb{P}_{x,h}(\tilde{P}_h(x) \neq P(x)) = \mathbb{E}_x \mathbb{E}_h \mathbb{1}_{\tilde{P}_h(x) \neq P(x)} \leq \sigma \beta_2,$$

and thus

$$\mathbb{P}_h \left[\mathbb{P}_x \left(\tilde{P}_h(x) \neq P(x) \right) \geq \sigma \right] \leq \beta_2.$$

Let $P_i \stackrel{\text{def}}{=} D_{h_i} P$, so that P_i is of degree $\leq d$ and \tilde{P}_h is a function of P_1, \dots, P_C . Now setting $\beta_1 := \frac{\beta}{2|\mathbb{F}|^2}$ and $\beta_2 := \frac{\beta}{2}$ finishes the proof. \blacksquare

3 Notions of Regularity

Notions of regularity similar to the following one defined by Green and Tao [GT09], play an important role in concepts related to polynomials.

Definition 3.1 (F -regularity) *Let \mathcal{F} be a factor of degree d , and let $F : \mathbb{N} \rightarrow \mathbb{N}$ be a growth function. We say that \mathcal{F} is F -regular if for every collection of coefficients $\{c_{i,j} \in \mathbb{F}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$, we have*

$$\text{rank}_{k-1} \left(\sum_{i=1}^d \sum_{j=1}^{M_i} c_{i,j} P_{i,j} \right) \geq F(\dim(\mathcal{F})),$$

where k is the largest i such that there is a nonzero $c_{i,j}$.

The following, by now well-known, lemma states that given a polynomial factor one can refine it to a regular one.

Lemma 3.2 (Regularity Lemma for Polynomials) *Let $d \geq 1$, $F : \mathbb{N} \rightarrow \mathbb{N}$ be a growth function, and let \mathcal{F} be a factor of degree d . Then there exists an F -regular extension \mathcal{F}' of \mathcal{F} of same degree d , satisfying*

$$\dim(\mathcal{F}') = O_{F,d,\dim(\mathcal{F})}(1).$$

We provide a proof for two reasons, self-containment and the fact that we will use the idea of this proof later in the section.

Proof: We shall induct on the dimension vectors (M_1, \dots, M_d) . Notice that the dimension vectors of a factor of degree d takes values in \mathbb{N}^d , and we will use the reverse lexicographical ordering on this space for our induction. The proof for $d = 1$ is obvious, because non-zero linear functions have infinite rank₀, unless there is a linear dependency between the polynomials in \mathcal{F} , which we can simply discard by removing the polynomials that can be written as a linear combination of the others.

If \mathcal{F} is already F -regular, then we are done. Otherwise, there is a set of coefficients $\{c_{i,j}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$, not all zero, such that

$$\text{rank}_{k-1} \left(\sum_{i=1}^d \sum_{j=1}^{M_i} c_{i,j} P_{i,j} \right) < F(\dim(\mathcal{F})).$$

Without loss of generality assume that $c_{k,M_k} \neq 0$. The above inequality means that P_{k,M_k} can be written as a function of the rest of the polynomials in the factor and a set of at most $F(\dim(\mathcal{F}))$ polynomials of degree at most $k - 1$. We will replace P_{k,M_k} with these new polynomials. The new factor will have the following dimension vector

$$(M_1, \dots, M_{k-1} + F(\dim(\mathcal{F})), M_k - 1, M_{k+1}, \dots, M_d).$$

Now by the induction hypothesis the above can be regularized. ■

The nature of the proof in regularity lemmas of the above type results in Ackermann like functions even for “reasonable” growth functions F .

3.1 Unbiased Factors

In what follows we suggest a new type of regularity which seems better suited for algorithmic reasons. The definition is inspired by the fact that in the applications of the regularity lemma one is usually interested in the linear combinations of the polynomials in the factor being unbiased. This definition will not suffice for our applications, but serves as a good starting point for stronger notions that we will introduce later.

Definition 3.3 (γ -unbiased factors) *Let \mathcal{F} be a factor of degree d , and let $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ be a decreasing function. We say that \mathcal{F} is γ -unbiased if for every collection of coefficients $\{c_{i,j} \in \mathbb{F}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$, we have*

$$\text{bias}\left(\sum_{i,j} c_{i,j} P_{i,j}\right) \leq \gamma(\dim(\mathcal{F})).$$

The following can be thought of as an algorithmic analogue of the regularity lemma.

Lemma 3.4 (Unbiased almost Refinement) *Let $d \geq 1$ be an integer. Suppose $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function and $\sigma, \rho \in (0, 1]$. There is a randomized algorithm that given a factor \mathcal{F} of degree d , runs in time $O_{\gamma, \rho, \sigma, \dim(\mathcal{F})}(n^d)$ and with probability $1 - \rho$ returns a γ -unbiased factor \mathcal{F}' with $\dim(\mathcal{F}') = O_{\gamma, \rho, \sigma, \dim(\mathcal{F})}(1)$, such that \mathcal{F}' is σ -close to being a refinement of \mathcal{F} .*

Proof: The proof idea is similar to that of [Lemma 3.2](#) in the sense that we do the same type of induction. The difference is that at each step we will have to control the errors that we introduce and the probability of correctness. At all steps in the proof without loss of generality we will assume that the polynomials in the factor are linearly independent, because otherwise we can always detect such a linear combination in $O_{\gamma, \rho, \sigma, \dim(\mathcal{F})}(n^d)$ time and remove a polynomial that can be written as a linear combination of the rest of the polynomials in the factor.

The base case for $d = 1$ is simple, a linearly independent set of non-constant linear polynomials is not biased at all, namely it is 0-unbiased. Assume that \mathcal{F} is γ -biased, then there exists a set of coefficients $\{c_{i,j} \in \mathbb{F}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$ such that

$$\text{bias}\left(\sum_{i,j} c_{i,j} P_{i,j}\right) \geq \gamma(\dim(\mathcal{F})).$$

To detect this, we will use the following algorithm:

We will estimate bias of each of the $|\mathbb{F}|^{\dim(\mathcal{F})}$ linear combinations and check whether it is greater than $\frac{3\gamma(\dim(\mathcal{F}))}{4}$. To do so, for each linear combination $\sum_{i,j} c_{i,j} P_{i,j}$ independently select a set of vectors x_1, \dots, x_C uniformly at random from \mathbb{F}^n , and let $\widetilde{\text{bias}}(\sum_{i,j} c_{i,j} P_{i,j}) \stackrel{\text{def}}{=} \left| \frac{1}{C} \sum_{\ell \in [C]} e_{\mathbb{F}}(y_{\ell}) \right|$, where $y_{\ell} = \sum_{i,j} c_{i,j} \cdot P_{i,j}(x_{\ell})$. Choosing $C = O_{\dim(\mathcal{F})} \left(\frac{1}{\gamma(\dim(\mathcal{F}))^2} \log\left(\frac{1}{\rho}\right) \right)$, we can distinguish bias $\geq \gamma$ from bias $\leq \frac{\gamma}{2}$, with probability $1 - \rho'$, where $\rho' := \frac{\rho}{4|\mathbb{F}|^{\dim(\mathcal{F})}}$. Let $\sum_{i,j} c_{i,j} P_{i,j}$ be such that the estimated bias was above $\frac{3\gamma(\dim(\mathcal{F}))}{4}$ and k be its degree. We will stop if there is no such linear combination or if the factor is of degree 1. Since by a union bound with probability at least $1 - \frac{\rho}{4}$, $\text{bias}(\sum_{i,j} c_{i,j} P_{i,j}) \geq \frac{\gamma(\dim(\mathcal{F}))}{2}$, by [Lemma 2.1](#) we can find, with probability $1 - \frac{\rho}{4}$, a set of polynomials Q_1, \dots, Q_r of degree $k - 1$ such that

- $\sum_{i,j} c_{i,j} P_{i,j}$ is $\frac{\sigma}{2}$ -close to a function of Q_1, \dots, Q_r ,

- $r \leq \frac{16|\mathbb{F}|^5}{\gamma(\dim(\mathcal{F}))^2 \cdot \sigma \cdot \rho}$.

We replace one polynomial of highest degree that appears in $\sum_{i,j} c_{i,j} P_{i,j}$ with polynomials Q_1, \dots, Q_r .

We will prove by the induction that our algorithm satisfies the statement of the lemma. For the base case, if \mathcal{F} is of degree 1, our algorithm does not refine \mathcal{F} by design. Notice that since we have removed all linear dependencies, \mathcal{F} is in fact 0-unbiased in this case.

Now given a factor \mathcal{F} , if \mathcal{F} is γ -biased, then with probability $1 - \rho'$ our algorithm will refine \mathcal{F} . With probability $1 - \frac{\rho}{4}$ the linear combination used for the refinement is $\frac{\gamma(\dim(\mathcal{F}))}{2}$ -biased. Let $\tilde{\mathcal{F}}$ be the outcome of one step of our algorithm. With probability $1 - \frac{\rho}{4}$, $\tilde{\mathcal{F}}$ is $\frac{\sigma}{2}$ -close to being a refinement of \mathcal{F} . Using the induction hypothesis with parameters $\gamma, \frac{\sigma}{2}, \frac{\rho}{4}$ we can find, with probability $1 - \frac{\rho}{4}$, a γ -unbiased factor \mathcal{F}' which is $\frac{\sigma}{2}$ -close to being a refinement of $\tilde{\mathcal{F}}$ and therefore, with probability at least $1 - (\frac{\rho}{4} + \frac{\rho}{4} + \frac{\rho}{4} + \rho') > 1 - \rho$, is σ -close to being a refinement of \mathcal{F} . ■

One of the main reasons one is interested in regular factors is that they partition the space into almost equal atoms (See for example [KL08, TZ12, GT09]). We observe that the same is true for unbiased factors.

Lemma 3.5 (Equidistribution for unbiased factors) *Suppose that $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function. Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a γ -unbiased factor of degree $d > 0$. Suppose that $b \in \mathbb{F}^m$. Then*

$$\mathbb{P}_{x \in \mathbb{F}^n} [\mathcal{F}(x) = b] = \frac{1}{\|\mathcal{F}\|} \pm \gamma(\dim(\mathcal{F})),$$

where $\|\mathcal{F}\| = |\Sigma|$ is the number of atoms of \mathcal{F} .

Proof:

$$\begin{aligned} \mathbb{P}_{x \in \mathbb{F}^n} [\mathcal{F}(x) = b] &= \mathbb{E}_x \left[\prod_{i \in [m]} \frac{1}{|\mathbb{F}|} \sum_{\lambda_i=0}^{|\mathbb{F}|-1} e_{\mathbb{F}}(\lambda_i(P_i(x) - b_i)) \right] \\ &= \frac{1}{|\mathbb{F}|^m} \sum_{\lambda \in \mathbb{F}^m} \mathbb{E}_{x \in \mathbb{F}^n} \left[e_{\mathbb{F}} \left(\sum_{i \in [m]} \lambda_i (P_i(x) - b_i) \right) \right] \\ &= \frac{1}{|\mathbb{F}|^m} (1 \pm \gamma(\dim(\mathcal{F})) |\mathbb{F}|^m) = \frac{1}{\|\mathcal{F}\|} \pm \gamma(\dim(\mathcal{F})), \end{aligned}$$

where the last inequality follows from the fact that for every $\lambda \neq \underline{0}$, the polynomial $\sum_i \lambda_i (P_i(x) - b_i)$ is nonzero and therefore has bias less than $\gamma(\dim(\mathcal{F}))$. ■

Let $P : \mathbb{F}^n \rightarrow \mathbb{F}$ be a function, and let \mathcal{F} be a polynomial factor such that P is measurable in \mathcal{F} . This means that there exists a function $\Gamma : \mathbb{F}^{\dim(\mathcal{F})} \rightarrow \mathbb{F}$ such that $P(x) = \Gamma(\mathcal{F}(x))$. Although we know that such a Γ exists, but we do not have explicit description of Γ . It is a simple but useful observation that we can provide query access to Γ in case \mathcal{F} is unbiased. It is worth recording this as the following lemma.

Lemma 3.6 (Query access) *Suppose that $\beta \in (0, 1]$ and let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a γ -unbiased factor of degree d , where $\gamma : \mathbb{F}^n \rightarrow \mathbb{F}$ is a decreasing function which decreases suitably fast. Suppose*

that we are given query access to a function $f : \mathbb{F}^n \rightarrow \mathbb{F}$, where f is measurable in \mathcal{F} , namely there exists $\Gamma : \mathbb{F}^{\dim(\mathcal{F})} \rightarrow \mathbb{F}$ such that $f(x) = \Gamma(\mathcal{F}(x))$. There is a randomized algorithm that, takes as input a value $a = (a_1, \dots, a_m) \in \mathbb{F}^m$, makes a single query to f , runs in $O(n^d)$, and returns a value $y \in \mathbb{F}$ such that $\Gamma(a) = y$ with probability greater than $1 - \beta$.

Proof: Choose $\gamma(x) := \frac{1}{2^{p^x}}$. By [Lemma 3.5](#)

$$|\mathcal{F}^{-1}(a)| \geq \frac{1}{\|\mathcal{F}\|} - \frac{1}{2\|\mathcal{F}\|} = \frac{1}{2\|\mathcal{F}\|}.$$

Let x_1, \dots, x_K be chosen uniformly at random from \mathbb{F}^n , where $K \geq 2\|\mathcal{F}\| \log \frac{1}{\beta}$. Then with probability at least $1 - \beta$, there is $i^* \in [K]$ such that $\mathcal{F}(x_{i^*}) = (a_1, \dots, a_m)$. This means that $x_{i^*} \in \mathcal{F}^{-1}(a)$. Notice that we can find this i^* , since we have explicit description of polynomials in \mathcal{F} and we can evaluate them on each x_i . Our algorithm will query f on input x_{i^*} and return $f(x_{i^*})$. ■

3.2 Uniform Factors

As it turns out, one sometimes needs stronger equidistribution properties from a factor. For example it is sometimes important to have an almost independence property, similar to [Lemma 3.5](#), for the factor on a random k -dimensional parallelepiped. Although this holds for a regular factor (See [\[GT09\]](#) Proposition 4.5), our notion of unbiased factors fails to satisfy such properties. To this end we define the following notion of a uniform factor.

Definition 3.7 (γ -uniform factors) Let \mathcal{F} be a factor of degree d , and let $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ be a decreasing function. We say that \mathcal{F} is γ -uniform if for every collection of coefficients $\{c_{i,j} \in \mathbb{F}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$ with $\deg\left(\sum_{i,j} c_{i,j} P_{i,j}\right) = k$,

$$\left\| e_{\mathbb{F}}\left(\sum_{i,j} c_{i,j} P_{i,j}\right) \right\|_{U^k} \leq \gamma(\dim(\mathcal{F})).$$

Remark 3.8 Notice that every γ -uniform factor is also γ -unbiased since

$$|\mathbb{E}[f(x)]| \leq \|f\|_{U^k},$$

and indeed uniformity is a stronger notion of regularity.

In [Section 4](#) we will present an algorithm to refine a given factor to a uniform one. We will need an assumption that $|\mathbb{F}|$ is larger than the degree of the factor. The reason is that our proof uses division by $d!$ where d is the degree of the factor, and therefore we will need $|\mathbb{F}|$ to be greater than d .

3.3 Strongly Unbiased Factors

As mentioned in previous section, notions of regularity ([Definition 3.1](#)) and uniformity ([Definition 3.7](#)) fail to address the case when the field \mathbb{F} has small characteristic. Kaufman and Lovett [\[KL08\]](#) introduce a stronger notion of regularity to prove the “bias implies low rank” in the general case. To define their notion of regularity we first have to define the derivative space of

a factor. During this section, for simplicity in the writing, we will denote a factor \mathcal{F} by a set of its polynomials $\{P_1, \dots, P_m\}$ where $m = \dim(\mathcal{F})$. In this notation, we no longer use the index i to denote the degree of P_i .

Definition 3.9 (Derivative Space) Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a polynomial factor over \mathbb{F}^n . Recall that

$$D_h P(x) = P(x+h) - P(x),$$

is the additive derivative of P in direction h , where h is a vector from \mathbb{F}^n . We define

$$\text{Der}(\mathcal{F}) \stackrel{\text{def}}{=} \{(D_h P_i)(x) : i \in [m], h \in \mathbb{F}^n\},$$

as the derivative space of \mathcal{F} .

Definition 3.10 (Strong regularity of polynomials [KL08]) Suppose that $F : \mathbb{N} \rightarrow \mathbb{N}$ is an increasing function. Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a polynomial factor of degree d and $\Delta : \mathcal{F} \rightarrow \mathbb{N}$ assigns a natural number to each polynomial in \mathcal{F} . We say that \mathcal{F} is strongly F -regular with respect to Δ if

1. For every $i \in [m]$, $1 \leq \Delta(P_i) \leq \deg(P_i)$.
2. For any $i \in [m]$ and $r > \Delta(P_i)$, there exist a function $\Gamma_{i,r}$ such that

$$P_i(x + y_{[r]}) = \Gamma_{i,r}(P_j(x + y_J) : j \in [m], J \subseteq [r], |J| \leq \Delta(P_j)),$$

where x, y_1, \dots, y_r are variables in \mathbb{F}^n , and $y_J \stackrel{\text{def}}{=} \sum_{k \in J} y_k$ for every $J \subseteq [r]$,

3. For any $r \geq 0$, let $\mathcal{C} = \{c_{i,I} \in \mathbb{F}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ be a collection of field elements. Define

$$Q_{\mathcal{C}}(x, y_1, \dots, y_r) \stackrel{\text{def}}{=} \sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} \cdot P_i(X + Y_I).$$

Let $\mathcal{F}' \subseteq \mathcal{F}$ be the set of all P_i 's which appear in $Q_{\mathcal{C}}$. There do not exist polynomials $\tilde{P}_1, \dots, \tilde{P}_\ell \in \text{Der}(\mathcal{F}')$, and $I_1, \dots, I_\ell \subseteq [r]$, with $\ell \leq F(\dim(\mathcal{F}))$, for which $Q_{\mathcal{C}}$ can be expressed as

$$\Gamma(\tilde{P}_1(x + y_{I_1}), \dots, \tilde{P}_\ell(x + y_{I_\ell})),$$

for some function $\Gamma : \mathbb{F}^\ell \rightarrow \mathbb{F}$.

The following lemma states that every polynomial factor can be refined to a strongly regular factor.

Lemma 3.11 (Strong-Regularity Lemma, [KL08]) Let $F : \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function. Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a polynomial factor of degree d . There exists a refinement $\mathcal{F}' = \{P_1, \dots, P_r\}$ of same degree d along with a function $\Delta : \mathcal{F}' \rightarrow \mathbb{N}$ such that

- \mathcal{F}' is strongly F -regular with respect to Δ ,
- $\dim(\mathcal{F}') = O_{F, \dim(\mathcal{F}), d}(1)$.

Moreover they prove that if a polynomial is approximated by a strongly regular factor, then it is in fact measurable with respect to the factor.

Lemma 3.12 ([KL08]) For every d there exists a constant $\varepsilon_d = 2^{-\Omega(d)}$ such that the following holds. Let $P : \mathbb{F}^n \rightarrow \mathbb{F}$ be a polynomial of degree d , P_1, \dots, P_s be a strongly regular collection of polynomials of degree $d-1$ with degree bound Δ and $\Gamma : \mathbb{F}^s \rightarrow \mathbb{F}$ be a function such that $\Gamma(P_1, \dots, P_s)$ is ε_d -close to P . Then there exists $\Gamma' : \mathbb{F}^s \rightarrow \mathbb{F}$ such that

$$P(x) = \Gamma'(P_1(x), \dots, P_s(x)).$$

Lemma 3.11 and **Lemma 3.12** combined with Bogdanov-Viola Lemma immediately gives the following theorem.

Theorem 3.13 (Bias implies low rank for general fields [KL08]) Let $P : \mathbb{F}^n \rightarrow \mathbb{F}$ be a polynomial of degree d such that $\text{bias}(P) \geq \delta > 0$. Then $\text{rank}_{d-1} \leq c(d, \delta, \mathbb{F})$.

In **Section 6** we will give an algorithmic version of this theorem. To do this, as in previous sections, we will have to use a notion of regularity which is well suited for algorithmic applications. Uniform factors (**Definition 3.7**) fail to address fields of low characteristics, for the reason that to refine a factor to a uniform factor we make use of division by $d!$ which is not possible in fields with $|\mathbb{F}| \leq d$. Strong regularity of [KL08] suggests how to extend the notion of unbiased factors to a stronger version in quite a straight forward fashion.

Definition 3.14 (Strongly γ -unbiased factors) Suppose that $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function. Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a polynomial factor of degree d over \mathbb{F}^n and let $\Delta : \mathcal{F} \rightarrow \mathbb{N}$ assign a natural number to each polynomial in the factor. We say that \mathcal{F} is strongly γ -unbiased with degree bound Δ if

1. For every $i \in [m]$, $1 \leq \Delta(P_i) \leq \deg(P_i)$.
2. For every $i \in [m]$ and $r > \Delta(P_i)$, there exists a function $\Gamma_{i,r}$ such that

$$P_i(x + y_{[r]}) = \Gamma_{i,r}(P_j(x + y_J) : j \in [m], J \subseteq [r], |J| \leq \Delta(P_j)).$$

3. For any $r \leq 0$ and not all zero collection of field elements $\mathcal{C} = \{c_{i,I} \in \mathbb{F}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ we have

$$\text{bias} \left(\sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} \cdot P_i(x + y_I) \right) \leq \gamma(\dim(\mathcal{F})),$$

for random variables $x, y_1, \dots, y_r \in \mathbb{F}^n$.

We will say \mathcal{F} is Δ -bounded if it only satisfies (1) and (2).

In **Section 6** we will show how to refine a given factor to a strongly unbiased one. This will allow us prove an algorithmic analogue of **Lemma 3.12**, which will be presented in **Section 6.1**.

4 Uniform Refinement

In this section we shall assume that the field \mathbb{F} has large characteristic, namely $|\mathbb{F}|$ is greater than the degree of the polynomials and the degree of the factors that we study. We will prove the following lemma which states that we can efficiently refine a given factor to a uniform factor.

Lemma 4.1 (Uniform Refinement) *Suppose $d < |\mathbb{F}|$ is a positive integer and $\rho \in (0, 1]$ is a parameter. There is a randomized algorithm that, takes as input a factor \mathcal{F} of degree d over \mathbb{F}^n , and a decreasing function $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$, runs in time $O_{\rho, \gamma, \dim(\mathcal{F})}(n^d)$, and with probability $1 - \rho$ outputs a γ -uniform factor $\tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is a refinement of \mathcal{F} , is of same degree d , and $|\dim(\tilde{\mathcal{F}})| \ll_{\sigma, \gamma, \dim(\mathcal{F})} 1$.*

Note 4.2 *Notice that unlike Lemma 3.4, here \mathcal{F}' is an exact refinement of \mathcal{F} . This will be achieved by the use of Proposition 4.4 which roughly states that approximation of a polynomial by a uniform factor implies its exact computation.*

The following proposition of Green and Tao states that if a polynomial is approximated well enough by a regular factor, then it is computed by the factor.

Proposition 4.3 ([GT09]) *Suppose that $d \geq 1$ is an integer. There exists a constant $\sigma_d > 0$ such that the following holds. Suppose that $P : \mathbb{F}^n \rightarrow \mathbb{F}$ is a polynomial of degree d and that \mathcal{F} is an F -regular factor of degree $d - 1$ for some increasing function $F : \mathbb{N} \rightarrow \mathbb{N}$ which increases suitably rapidly in terms of d . Suppose that $\tilde{P} : \mathbb{F}^n \rightarrow \mathbb{F}$ is an \mathcal{F} -measurable function and that $\mathbb{P}(P(x) \neq \tilde{P}(x)) \leq \sigma_d$. Then P is itself \mathcal{F} -measurable.*

Our proof of Lemma 4.1 will require the following variant of Proposition 4.3, the proof of which will be deferred to Sections 4.1 and 4.2. This proposition shows that the uniformity notion is strong enough to imply results from [GT09] such as Proposition 4.3.

Proposition 4.4 *Suppose that $d \geq 1$ is an integer. There exists $\sigma_{4.4}(d) > 0$ such that the following holds. Suppose that $P : \mathbb{F}^n \rightarrow \mathbb{F}$ is a polynomial of degree d and that \mathcal{F} is a $\gamma_{4.4}$ -uniform factor of degree $d - 1$ for some decreasing function $\gamma_{4.4}$ which decreases suitably rapidly in terms of d . Suppose that $\tilde{P} : \mathbb{F}^n \rightarrow \mathbb{F}$ is an \mathcal{F} -measurable function and that $\mathbb{P}(P(x) \neq \tilde{P}(x)) \leq \sigma_{4.4}(d)$. Then P is itself \mathcal{F} -measurable.*

Lemma 4.1 and Proposition 4.4 are somewhat entangled: We will prove Lemma 4.1 by induction on the dimension vector of \mathcal{F} . For the induction step, we check whether there is a linear combination of polynomials in \mathcal{F} that has large Gowers norm. We will use this to replace a polynomial P from \mathcal{F} with a set of lower degree polynomials. To do this, we first approximate P with a few lower degree polynomials Q_1, \dots, Q_r , then use the induction hypothesis to refine $\{Q_1, \dots, Q_r\}$ to a uniform factor $\{\tilde{Q}_1, \dots, \tilde{Q}_{r'}\}$ and use Proposition 4.4 to conclude that P is measurable in $\{\tilde{Q}_1, \dots, \tilde{Q}_{r'}\}$.

Proof of Lemma 4.1: Let \mathcal{F} be a factor of degree d . Similar to the proof of Lemma 3.4 we will induct on dimension vector (M_1, \dots, M_d) . At all steps of the proof we may assume that all polynomials in \mathcal{F} are homogeneous, that is, all multinomials are of same degree. This is because otherwise, we may replace each polynomial $P_{i,j}$ with i homogeneous polynomials that sum up to $P_{i,j}$. Moreover, as in the proof of Lemma 3.4, without loss of generality we may assume that the polynomials in \mathcal{F} are linearly independent, since otherwise we can detect and remove linear dependencies in $O_{\dim(\mathcal{F})}(n^d)$ time.

Given the above assumptions, the base case for $d = 1$ is simple, a linearly independent homogeneous factor is 0-uniform. Let $d > 1$ and assume that \mathcal{F} is not γ -uniform, then there exists a set of coefficients $\{c_{i,j} \in \mathbb{F}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$ such that

$$\left\| e_{\mathbb{F}} \left(\sum_{i,j} c_{i,j} \cdot P_{i,j} \right) \right\|_{U^k} \geq \gamma(\dim(\mathcal{F})),$$

where $k = \deg(\sum_{i,j} c_{i,j} \cdot P_{i,j})$. We will use the following randomized algorithm to detect such a linear combination:

We will estimate $\left\| e_{\mathbb{F}}(\sum_{i,j} c_{i,j} \cdot P_{i,j}) \right\|_{U^k}^{2^k}$ for each of the $|\mathbb{F}|^{\dim(\mathcal{F})}$ linear combinations and check whether it is greater than $\frac{3\gamma(\dim(\mathcal{F}))^{2^k}}{4}$. To do this, as in the proof of [Lemma 3.4](#), independently for each linear combination $\sum_{i,j} c_{i,j} \cdot P_{i,j}$ select C sets of vectors $x^{(\ell)}, y_1^{(\ell)}, \dots, y_k^{(\ell)}$ chosen uniformly at random from \mathbb{F}^n for $1 \leq \ell \leq C$, and compute

$$\left| \sum_{\ell \in [C]} e_{\mathbb{F}} \left(\sum_{I \subseteq [k]} (-1)^{|I|} \cdot \sum_{i,j} c_{i,j} \cdot P_{i,j}(x^{(\ell)} + y_I^{(\ell)}) \right) \right|$$

Choosing $C = O_{\dim(\mathcal{F})} \left(\frac{1}{\gamma(\dim(\mathcal{F}))^2} \log\left(\frac{1}{\rho}\right) \right)$, we can distinguish between Gowers norm $\leq \frac{\gamma(\mathcal{F})}{2^{1/2^k}}$ and Gowers norm $\geq \gamma(\mathcal{F})$, with $\rho' = \frac{\rho}{4^{|\mathbb{F}|^{\dim(\mathcal{F})}}}$ probability of error.

Let $Q \stackrel{\text{def}}{=} \sum_{i,j} c_{i,j} \cdot P_{i,j}$, with $\deg(Q) = k$, the detected linear combination for which the estimated $\|e_{\mathbb{F}}(Q)\|_{U^k}^{2^k}$ is greater than $\frac{3\gamma(\dim(\mathcal{F}))^{2^k}}{4}$. By a union bound on all the linear combinations, with probability at least $1 - \frac{\rho}{4}$, $\|e_{\mathbb{F}}(Q)\|_{U^k}^{2^k} \geq \frac{\gamma(\dim(\mathcal{F}))^{2^k}}{2}$. Notice that

$$\text{bias}(DQ) = \|e(Q)\|_{U^k}^{2^k} \geq \frac{\gamma(\dim(\mathcal{F}))}{2},$$

where $DQ : (\mathbb{F}^n)^k \rightarrow \mathbb{F}$ is defined as

$$DQ(y_1, \dots, y_k) \stackrel{\text{def}}{=} D_{y_1} D_{y_2} \cdots D_{y_k} Q(x).$$

Let $\tilde{\sigma}$ and $\tilde{\gamma}$ be as in [Proposition 4.4](#). By [Lemma 2.1](#) we can find, with probability $1 - \frac{\rho}{4}$, a set of polynomials Q_1, \dots, Q_r of degree $k - 1$ such that

- DQ is $\tilde{\sigma}$ -close to being measurable with respect to Q_1, \dots, Q_r .
- $r \leq \frac{8|\mathbb{F}|^5}{\gamma(\dim(\mathcal{F}))^{2^{k+1}} \tilde{\sigma} \rho}$.

By the induction hypothesis, with probability $1 - \frac{\rho}{4}$, we can refine $\{Q_1, \dots, Q_r\}$ to a new factor $\{\tilde{Q}_1, \dots, \tilde{Q}_{r'}\}$, which is $\tilde{\gamma}$ -uniform. It follows from [Proposition 4.4](#) that DQ is in fact measurable in $\{\tilde{Q}_1, \dots, \tilde{Q}_{r'}\}$. Since $|\mathbb{F}| > k$ we can write the following Taylor expansion

$$Q(x) = \frac{DQ(x, \dots, x)}{k!} + R(x), \tag{4}$$

where $R(x)$ is a polynomial of degree $\leq k - 1$ which can be computed from Q . Thus we can replace a maximum degree polynomial $P_{i,j}$, that appears in Q , with polynomials $\tilde{Q}_1, \dots, \tilde{Q}_{r'}$, and $R(x)$. The proof follows by using the induction hypothesis for parameters $\frac{\rho}{4}$ and γ . \blacksquare

4.1 Equidistribution over parallelepipeds

Since γ -uniform factors are also γ -unbiased, they automatically inherit the equidistribution property, [Lemma 3.5](#). In this section we shall prove an equidistribution property which was the reason we introduced uniform factors in the first place. To this end we will need the following lemma.

Lemma 4.5 (Near orthogonality) For every decreasing function $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}^+$ and parameter $d \geq 1$, there is a decreasing function $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ such that the following holds. Suppose that $\mathcal{F} = \{P_{i,j}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$ is a γ -uniform factor of degree d . Let $x \in \mathbb{F}^n$ be a fixed vector. For every integer k and set of not all zero coefficients $\Lambda = \{\lambda_{i,j,\omega}\}_{1 \leq i \leq d, 1 \leq j \leq M_i, \omega \in \{0,1\}^k}$ define

$$P_\Lambda(y_1, \dots, y_k) \stackrel{\text{def}}{=} \sum_{i \in [d], j \in [M_i]} \sum_{\omega \in \{0,1\}^k} \lambda_{i,j,\omega} \cdot P_{i,j}(x + \omega \cdot y),$$

where $y = (y_1, \dots, y_k) \in (\mathbb{F}^n)^k$. Then either we have that $P_\Lambda \equiv 0$ or

$$\text{bias}(P_\Lambda) < \varepsilon(\dim(\mathcal{F}))$$

Note 4.6 Notice that the statement of this lemma does not require the characteristic $|\mathbb{F}|$ to be large. Such near orthogonality result over systems of linear forms was proved for the large characteristic case, $d < |\mathbb{F}|$, by Hatami and Lovett [HL11]. Bhattacharyya, et. al. [BFH⁺13] later proved such near orthogonality result for affine systems of linear forms without any assumption on the characteristic of the field. Our proof of Lemma 4.5 uses the derivative technique used in [BFH⁺13] except the fact that in our case the linear forms of interest are in $\{0,1\}^k$ makes the proof simpler.

Proof Lemma 4.5: For a vector $\omega \in \{0,1\}^d$, let $|\omega|$ be equal to the size of its support. We will say $\omega' \preceq \omega$ if $\omega'_i \leq \omega_i$ for every i . We may assume that for each i, j , and ω , $\deg(P_{i,j}) = i \geq |\omega|$, because otherwise we may use the identity

$$\sum_{\omega' \preceq \omega} (-1)^{|\omega'|} P_{i,j}(x + \omega' \cdot y) = 0,$$

to replace $P_{i,j}(x + \omega \cdot y)$ with $P_{i,j}(x + \omega' \cdot y)$'s where $|\omega'| < |\omega|$. Notice that $P_{\Lambda'} \equiv 0$ after this process is equivalent to $P_\Lambda \equiv 0$.

Let ω^* be of maximum support among all ω for which there is an i such that $\lambda_{i,\omega}$ is nonzero. Roughly speaking, we will derive P_Λ in a way that terms of the form $\lambda_{i,j,\omega^*} \cdot P_{i,j}$ will be the only terms that can survive. Without loss of generality assume that $|\omega^*| = r$ and that $\{1, \dots, r\}$ is the support of ω^* . We will consider

$$DP_\Lambda(y_1, \dots, y_k, h_1, \dots, h_r) \stackrel{\text{def}}{=} D_{h_1} D_{h_2} \cdots D_{h_r} P_\Lambda(y_1, \dots, y_d),$$

where $h_i \in (\mathbb{F}^n)^d$ is only supported on y_i 's entries. It follows from the maximality of ω^* that

$$DP_\Lambda(y_1, \dots, y_k, h_1, \dots, h_r) = \sum_{i \in [d], j \in [M_i]} \lambda_{i,j,\omega^*} \cdot D_{h'_1} \cdots D_{h'_r} P_{i,j}(x + \omega^* \cdot y),$$

where each h'_i is the restriction of h_i to y_i 's entries, and therefore by a simple change of variables we get

$$\text{bias}(DP_\Lambda) = \left\| e \left(\sum_{i,j} \lambda_{i,j,\omega^*} \cdot P_{i,j} \right) \right\|_{U^r}^{2^r}, \quad (5)$$

On the other hand we have the following claim which is a result of r repeated applications of Cauchy-Schwarz inequality

Claim 4.7

$$\text{bias}(P_\Lambda)^{2^r} \leq \text{bias}(DP_\Lambda).$$

Proof: It suffices to show that we have

$$\left| \mathbb{E}_{y_1, \dots, y_k, h \in \mathbb{F}^n} e_{\mathbb{F}}(P_\Lambda(y_1 + h, y_2, \dots, y_k) - P_\Lambda(y_1, y_2, \dots, y_k)) \right| \geq \left| \mathbb{E}_{y_1, \dots, y_k} e_{\mathbb{F}}(P_\Lambda(y_1, \dots, y_k)) \right|^2.$$

This is a simple application of Cauchy Schwarz inequality

$$\begin{aligned} \left| \mathbb{E}_{y_1, \dots, y_k, h} e_{\mathbb{F}}(P_\Lambda(y_1 + h, y_2, \dots, y_k) - P_\Lambda(y_1, \dots, y_k)) \right| &= \mathbb{E}_{y_2, \dots, y_k} \left| \mathbb{E}_z e_{\mathbb{F}}(P_\Lambda(z, y_2, \dots, y_k)) \right|^2 \\ &\geq \left| \mathbb{E}_{z, y_2, \dots, y_k} e_{\mathbb{F}}(P_\Lambda(z, y_2, \dots, y_k)) \right|^2. \end{aligned}$$

■

Now letting $\gamma(x) := \varepsilon(x)$, the proof follows by (5), the definition of γ -uniformity and Claim 4.7. ■

Let $\mathcal{F} = \{P_{i,j}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$ be a polynomial factor of degree d , and suppose we have a parallelepiped $(x + \omega \cdot y)_{\omega \in \{0,1\}^k}$, where $y = (y_1, \dots, y_k) \in (\mathbb{F}^n)^k$. The vector $(\mathcal{F}(x + \omega \cdot y))_{\omega \in \{0,1\}^k}$ can have some dependencies, and we cannot expect them to be uniformly distributed similar to Lemma 3.5. For example if P is a linear function then $\sum_{\omega \in \{0,1\}^2} P(x + \omega \cdot y) \equiv 0$ for every x . Therefore the image of \mathcal{F} over a parallelepiped of dimension k simply cannot take every possible value over \mathbb{F}^{nk} .

Consistence. Call a vector $b = (b_{i,j,\omega})_{1 \leq i \leq d, 1 \leq j \leq M_i, \omega \in \{0,1\}^k}$, where $b \in \mathbb{F}^{\dim(\mathcal{F})2^k}$, to be consistent with the factor \mathcal{F} if the following holds. For every set of coefficients $\Lambda = (\lambda_{i,j,\omega})_{1 \leq i \leq d, 1 \leq j \leq M_i, \omega \in \{0,1\}^k}$,

$$\sum_{i \in [d], j \in [M_i]} \sum_{\omega \in \{0,1\}^k} \lambda_{i,j,\omega} \cdot P_{i,j}(x + \omega \cdot y) \equiv 0,$$

implies

$$\sum_{i \in [d], j \in [M_i]} \sum_{\omega \in \{0,1\}^k} \lambda_{i,j,\omega} \cdot b_{i,j,\omega} = 0.$$

Let Σ_\square be the set of all vectors that are consistent with \mathcal{F} . It is easy to verify that Σ_\square forms a vector space. The following lemma of [GT09] computes the dimension of Σ_\square .

Lemma 4.8 ([GT09]) *Suppose that $k > d$. Then we have*

$$\dim(\Sigma_\square) = \sum_{i \in [d]} \sum_{0 \leq j \leq i} \binom{k}{j}.$$

Now the proof of the equidistribution over parallelepipeds follows in a straightforward manner.

Lemma 4.9 (Equidistribution of parallelepipeds) *Suppose that $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function, and $k > d > 0$ are integers. Suppose that $\mathcal{F} = \{P_{i,j}\}_{1 \leq i \leq d, 1 \leq j \leq M_i}$ is a $\gamma_{4.5}$ -uniform factor of degree d . Let $x \in \mathbb{F}^n$ be a fixed vector, $b = (b_{i,j,\omega})_{1 \leq i \leq d, 1 \leq j \leq M_i, \omega \in \{0,1\}^k} \in \mathbb{F}^{\dim(\mathcal{F})2^k}$ is consistent with \mathcal{F} , and moreover $P_{i,j}(x) = b_{i,j,\emptyset}$ for all i and j . Then we have*

$$\mathbb{P}_{x, y_1, \dots, y_k} \left[P_{i,j}(x + \omega \cdot y) = b_{i,j,\omega} \ \forall i \in [d], \forall j \in [M_i], \forall \omega \in \{0,1\}^k \right] = (1 \pm \varepsilon) \frac{1}{|\Sigma_\square|}.$$

Proof: Similar to the proof of [Lemma 3.5](#) we have

$$\begin{aligned}
\mathbb{P}_{y_1, \dots, y_k} [P_{i,j}(x + \omega \cdot y) = b_{i,j,\omega} \forall i, j, \omega] &= \mathbb{E} \left[\prod_{i,j,\omega} \frac{1}{|\mathbb{F}|} \left(\sum_{\lambda_{i,j,\omega}=0}^{|\mathbb{F}|-1} e_{\mathbb{F}} \left(\lambda_{i,j,\omega} (P_{i,j}(x + \omega \cdot y) - b_{i,j,\omega}) \right) \right) \right] \\
&= \frac{1}{|\mathbb{F}|^{\dim(\mathcal{F})2^k}} \sum_{\Lambda} e_{\mathbb{F}} \left(\sum_{i,j,\omega} \lambda_{i,j,\omega} b_{i,j,\omega} \right) \mathbb{E} \left[e_{\mathbb{F}} \left(\sum_{i,j,\omega} \lambda_{i,j,\omega} P_i(x + \omega \cdot y) \right) \right] \\
&= \frac{1}{|\Sigma_{\square}|} \pm \varepsilon(\dim(\mathcal{F})),
\end{aligned}$$

where i is always taken from $[d]$, j from $[M_i]$, ω from $\{0, 1\}^k$, and $\Lambda = (\lambda_{i,j,\omega})$ is a collection of coefficients from \mathbb{F} . The last inequality follows from [Lemma 4.5](#). \blacksquare

4.2 Proof of Proposition 4.4

Here we will present how to adapt the proof of Proposition 5.1 from [\[GT09\]](#) to our setting.

Proof of Proposition 4.4: Let X be the set of points in \mathbb{F}^n for which $P(x) = \tilde{P}(x)$. By assumption we know that $|X| \geq (1 - \sigma_d)|\mathbb{F}|^n$. We will use $(d + 1)$ -dimensional parallelepipeds to prove that $P(x) = \tilde{P}(x)$ within the whole of most of the atoms of \mathcal{F} , and then to prove that P should be constant on each remaining atom as well.

By [Lemma 3.5](#) we have that for $1 - O(\sqrt{\sigma_d})$ of the atoms A of \mathcal{F} we have $\mathbb{P}_{x \in A} (P(x) = \tilde{P}(x)) \geq 1 - O(\sqrt{\sigma_d})$; we will say that these atoms are *almost good*. We will prove that almost good atoms are in fact, totally good.

Suppose that $A = \mathcal{F}^{-1}(t)$ is an almost good atom, where $t = (t_{i,j})_{1 \leq i \leq d, 1 \leq j \leq M_i}$ is a vector in $\mathbb{F}^{\dim(\mathcal{F})}$. Let $A' \stackrel{\text{def}}{=} A \cap X$ be the set of points in A for which $P(x) = \tilde{P}(x)$. The following lemma proved in [\[GT09\]](#) can be adapted to our setting by [Lemma 3.5](#) and [Lemma 4.9](#).

Lemma 4.10 (Lemma 5.2 of [\[GT09\]](#)) *Suppose that σ_d is a sufficiently small constant. Fix an $x \in A$. Then there is h so that all of the vertices $x + \omega \cdot h$, $\omega \neq 0^{d+1}$, lie in A' .*

Let x be an arbitrary point in A , and let h be as in [Lemma 4.10](#). Since P is a degree d polynomial, we have the following derivative identity

$$P(x) = - \sum_{\omega \in \{0,1\}^{d+1}, \omega \neq 0} (-1)^{|\omega|} P(x + \omega \cdot h) = - \sum_{\omega \in \{0,1\}^{d+1}, \omega \neq 0} (-1)^{|\omega|} \tilde{P}(x + \omega \cdot h).$$

Since \tilde{P} is constant on A , this means that $P(x) = \tilde{P}(x)$. Since x was an arbitrary element of A , this implies that P is constant on A .

We have proved that $1 - O(\sqrt{\sigma_d})$ of the atoms of \mathcal{F} are totally good. Let A be an arbitrary atom of \mathcal{F} . The following claim will allow us to show that A is totally good.

Claim 4.11 *Assume that $1 - O(\sqrt{\sigma_d})$ of the atoms are totally good, where $\sigma_d \leq c2^{-d}$. Then for every atom A , there is a set of good atoms A_{ω} for $\omega \neq 0$ in $\{0, 1\}^{d+1}$ such that for every $x \in A$ there exists $h = (h_1, \dots, h_{d+1})$ such that $x + \omega \cdot h \in A_{\omega}$.*

Proof: Notice that by [Lemma 4.9](#) it suffices to prove this fact for one arbitrary point x in A . We will pick h_1, \dots, h_k uniformly at random from \mathbb{F}^n . By [Lemma 3.5](#), for each $\omega \neq 0$, the probability that $x + \omega \cdot h$ lies in a good atom is $1 - O(\sqrt{\sigma_d})$. Choosing $\sigma_d \leq c2^{-d}$ for sufficiently small constant c , it follows by union bound that with positive probability there is a choice of $h = (h_1, \dots, h_{d+1})$ such that $x + \omega \cdot h$ is a good atom. ■

Existence of good atoms A_ω as above implies that P has to be constant on A as well, again using the fact that P is a degree d polynomial and therefore

$$P(x) = - \sum_{\omega \in \{0,1\}^{d+1}, \omega \neq 0} (-1)^{|\omega|} P(x + \omega \cdot h).$$

This completes the proof of [Proposition 4.4](#). ■

5 Applications in High Characteristics

In this section we show some applications of the uniformity notion, and our algorithm for uniform refinement of a factor. Throughout the section we shall assume that the field \mathbb{F} has large characteristic, namely $|\mathbb{F}|$ is greater than the degree of the polynomials and the degree of the factors that we study.

5.1 Computing Polynomials with Large Gowers Norm

Following is an immediate corollary of [Lemma 2.1](#), [Lemma 3.6](#), and [Proposition 4.4](#).

Lemma 5.1 *Suppose that an integer d satisfies $0 \leq d < |\mathbb{F}|$. Let $\delta, \sigma, \beta \in (0, 1]$. There is a randomized algorithm that given query access to a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree d such that $\text{bias}(P) \geq \delta > 0$, runs in $O_{\delta, \beta, \sigma}(n^d)$ and with probability $1 - \beta$, returns a polynomial factor $\mathcal{F} = \{P_{i,j}\}_{1 \leq i \leq d-1, 1 \leq j \leq M_i}$ of degree $d-1$ and $\dim(\mathcal{F}) = O_{\delta, \beta, \sigma}(1)$ such that P is measurable in \mathcal{F} , namely there is a function Γ such that*

$$P(x) = \Gamma \left((P_{i,j}(x))_{1 \leq i \leq d-1, 1 \leq j \leq M_i} \right).$$

Moreover for every query to the value of $\Gamma(\cdot)$ for a vector from $\mathbb{F}^{\dim(\mathcal{F})}$ the algorithm returns the correct answer with probability $1 - \beta$.

Proof: Let σ_d and γ_1 be as in [Proposition 4.4](#), and let γ_2 be as in [Lemma 3.6](#). Set $\gamma_d := \min\{\gamma_1, \gamma_2\}$. By [Lemma 2.1](#) since $\text{bias}(P) \geq \delta$, in time $O_{\sigma_d, \beta}(n^d)$, with probability $1 - \frac{\beta}{2}$ we can find a polynomial factor \mathcal{F} of degree $d-1$ such that

- P is $\frac{\sigma_d}{2}$ -close to a function of \mathcal{F}
- $\dim(\mathcal{F}) \leq \frac{4|\mathbb{F}|^5}{\delta^2 \sigma_d \beta}$

By [Lemma 4.1](#), with probability $1 - \frac{\beta}{2}$, we can find a γ_d -uniform factor \mathcal{F}' such that \mathcal{F}' is $\frac{\sigma_d}{2}$ -close to being a refinement of \mathcal{F} .

Thus with probability greater than $1 - \beta$, P is σ_d -close to a function of the strongly γ_d -unbiased factor \mathcal{F}' , and by [Lemma 6.3](#) P is measurable in \mathcal{F}' . The query access to Γ can be granted by [Lemma 3.6](#). ■

The following is an algorithmic version of Proposition 6.1 of [GT09], which states that given a polynomial P such that $e_{\mathbb{F}}(P)$ has large Gowers norm, then we can compute P by a few lower degree polynomials.

Proposition 5.2 (Computing Polynomials with High Gowers Norm) *Suppose that $|\mathbb{F}| > d \geq 2$ and that $\delta, \beta \in (0, 1]$. There is a randomized algorithm that given query access to a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree d with $\|e_{\mathbb{F}}(P(x))\|_{U^d} \geq \delta$, runs in $O_{\delta, \beta}(n^d)$ and with probability $1 - \beta$, returns a polynomial factor \mathcal{F} of degree $d - 1$ such that*

- There is a function $\Gamma : \mathbb{F}^{\dim(\mathcal{F})} \rightarrow \mathbb{F}$ such that $P = \Gamma(\mathcal{F})$.
- $\dim(\mathcal{F}) = O_{d, \delta, \beta}(1)$.

Moreover, for every query to the value of $\Gamma(\cdot)$ for a vector from $\mathbb{F}^{\dim(\mathcal{F})}$, the algorithm returns the correct answer with probability $1 - \beta$.

Proof: Write $\partial^d P(h_1, \dots, h_d) := D_{h_1} \cdots D_{h_d} P(x)$. Since P has degree d , $\partial^d P$ does not depend on x . From the definition of the U^d norm, we have

$$\text{bias}(\partial^d P) = \|e(P)\|_{U^d}^{2^d} \geq \delta^{2^d}.$$

Applying Lemma 5.1 to $\partial^d P$, with probability $1 - \frac{\beta}{2}$, we can find a factor $\tilde{\mathcal{F}}$ of degree $d - 1$, such that $\dim(\tilde{\mathcal{F}}) = O_{\delta, \beta, d}(1)$ and $\partial^d P$ is measurable in $\tilde{\mathcal{F}}$.

It is easy to check that since $|\mathbb{F}| > d$, we have the following Taylor expansion

$$P(x) = \frac{1}{d!} \partial^d P(x, \dots, x) + Q(x),$$

where Q is a polynomial of degree $\leq d - 1$. We can find explicit description of P , and therefore Q in $O(n^d)$. Let $\mathcal{F}' := \tilde{\mathcal{F}} \cup \{Q\}$, and let γ be as in Lemma 3.6 for error parameter β . By Lemma 4.1, with probability $1 - \frac{\beta}{2}$, we can refine \mathcal{F}' to a γ -uniform factor \mathcal{F} of same degree $d - 1$ with $\dim(\mathcal{F}) = O_{\dim(\mathcal{F}'), \beta, \gamma}(1)$. Now query access can be granted by Lemma 3.6. ■

5.2 Decoding Reed-Muller Codes with Structured Noise

In this section we discuss the task of decoding Reed-Muller codes when the noise is “structured”. Recall that the codewords in a Reed-Muller code of order k , correspond to evaluations of all degree k polynomials in n variables over \mathbb{F} . We will consider the task of decoding codewords when the noise is structured in the sense that the applied noise to the codeword itself is a polynomial of higher degree d . In other words, we are given a polynomial P of degree $d > k$ with the promise that it is “close” to an unknown degree k polynomial, and the task is to find a degree k polynomial Q that is somewhat close to P . The assumption of $d > k$ is made, because the case $d \leq k$ is trivial.

Theorem 5.3 (Reed-Muller Decoding) *Suppose that $d > k > 0$ are integers, and let $\varepsilon \in (0, 1]$ and $\beta \in (0, 1]$ be parameters. There is $\delta(\varepsilon, d, k) > 0$ and a randomized algorithm that, given query access to a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree d , with the promise that there exists a polynomial Q of degree k such that $\mathbb{P}(P(x) = Q(x)) \geq \frac{1}{|\mathbb{F}|} + \varepsilon$, runs in $O(n^d)$, and with probability $1 - \beta$ returns a polynomial \tilde{Q} of degree k such that*

$$\mathbb{P}(P(x) \neq \tilde{Q}(x)) \geq \frac{1}{|\mathbb{F}|} + \delta.$$

Before presenting the proof of [Theorem 5.3](#) we will look at the relations between Hamming distance of P and Q , correlation between $e_{\mathbb{F}}(P)$ and $e_{\mathbb{F}}(Q)$, and Gowers norm of $e_{\mathbb{F}}(P)$. The following claim relates the Hamming distance of two polynomials P and Q to the correlation between $e_{\mathbb{F}}(P)$ and $e_{\mathbb{F}}(Q)$.

Claim 5.4 *Suppose that $\varepsilon \in (0, 1]$, and let $P, Q : \mathbb{F}^n \rightarrow \mathbb{F}$ be two functions such that $\mathbb{P}_x(P(x) = Q(x)) \geq 1/|\mathbb{F}| + \varepsilon$. Then there exists a nonzero $t \in \mathbb{F}$ for which*

$$|\langle e_{\mathbb{F}}(t \cdot P), e_{\mathbb{F}}(t \cdot Q) \rangle| = \left| \mathbb{E}_x e_{\mathbb{F}}(t \cdot (P(x) - Q(x))) \right| \geq \varepsilon.$$

Proof: We have

$$\begin{aligned} \frac{1}{|\mathbb{F}|} + \varepsilon &\leq \mathbb{P}_{x \in \mathbb{F}^n} (P(x) = Q(x)) = \mathbb{E}_{x \in \mathbb{F}^n} [1_{P(x)=Q(x)}] \\ &= \mathbb{E}_{x \in \mathbb{F}^n} \left[\frac{1}{|\mathbb{F}|} \sum_{t \in \mathbb{F}} e_{\mathbb{F}}(t \cdot (P(x) - Q(x))) \right] \\ &= \frac{1}{|\mathbb{F}|} \sum_{c \in \mathbb{F}} \langle e_{\mathbb{F}}(t \cdot P), e_{\mathbb{F}}(t \cdot Q) \rangle \\ &= \frac{1}{|\mathbb{F}|} \left(1 + \sum_{t \in \mathbb{F} \setminus \{0\}} \langle e_{\mathbb{F}}(t \cdot P), e_{\mathbb{F}}(t \cdot Q) \rangle \right), \end{aligned}$$

and thus there is $t \in \mathbb{F} \setminus \{0\}$ for which $|\langle e_{\mathbb{F}}(t \cdot P), e_{\mathbb{F}}(t \cdot Q) \rangle| \geq \varepsilon$. ■

The following lemma which was first proved in [[Gow01](#)] is a key property of Gowers norms.

Lemma 5.5 (Gowers Cauchy-Schwarz) *Let G be a finite Abelian group, and consider a family of functions $f_S : G \rightarrow \mathbb{C}$, where $S \subseteq [k]$. Then*

$$\left| \mathbb{E}_{X, Y_1, \dots, Y_d \in \mathbb{F}^n} \left[\prod_{S \subseteq [d]} c^{d-|S|} f_S(X + \sum_{i \in S} Y_i) \right] \right| \leq \prod_{S \subseteq [d]} \|f_S\|_{U^d}, \quad (6)$$

Assume that $P : \mathbb{F}^n \rightarrow \mathbb{F}$ is a polynomial of degree d and $Q : \mathbb{F}^n \rightarrow \mathbb{F}$ is a polynomial of degree $k < d$. Using the above lemma with $f_{\emptyset} := e_{\mathbb{F}}(P - Q)$ and $f_S := 1$ for $S \neq \emptyset$ implies that

$$|\langle e_{\mathbb{F}}(P), e_{\mathbb{F}}(Q) \rangle| = \left| \mathbb{E}_{x \in \mathbb{F}^n} e_{\mathbb{F}}(P(x) - Q(x)) \right| \leq \|e_{\mathbb{F}}(P(x) - Q(x))\|_{U^d} = \|e_{\mathbb{F}}(P(x))\|_{U^d}, \quad (7)$$

where the last equality follows from the fact that $\deg(Q(x)) = k < d$.

Proof of [Theorem 5.3](#): Since there exists a polynomial Q of degree k with $\mathbb{P}(P(x) \neq Q(x)) \leq \varepsilon$, by [Claim 5.4](#) there exists a nonzero $t \in \mathbb{F}$ such that

$$|\langle e_{\mathbb{F}}(t \cdot P), e_{\mathbb{F}}(t \cdot Q) \rangle| \geq \varepsilon.$$

Notice that

$$\|e_{\mathbb{F}}(t \cdot P)\|_{U^d} \geq \langle e_{\mathbb{F}}(t \cdot P), e_{\mathbb{F}}(t \cdot Q) \rangle \geq \varepsilon,$$

where the first inequality follows from (7) since $\deg(t \cdot Q) \leq k$. Notice that we may find such a constant $t \in \mathbb{F} \setminus \{0\}$ with high probability by estimating the U^d norm of tP for all $|\mathbb{F}| - 1$ possible choices of t . Set $\tilde{P} := t \cdot P$. By [Proposition 5.2](#), with probability $1 - \frac{\beta}{6}$ we can find a polynomial factor $\tilde{\mathcal{F}}$ of degree $d - 1$ such that \tilde{P} is measurable in $\tilde{\mathcal{F}}$. Let $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ be a decreasing function to be specified later. By [Lemma 4.1](#), with probability $1 - \frac{\beta}{6}$, we can refine $\tilde{\mathcal{F}}$ to a γ -uniform factor $\mathcal{F} = \{P_1, \dots, P_m\}$ of same degree $d - 1$, with $\dim(\mathcal{F}) = O_{\gamma, \beta}(1)$. Since \tilde{P} is measurable in \mathcal{F} , there exists $\Gamma : \mathbb{F}^{\dim(\mathcal{F})} \rightarrow \mathbb{F}$ such that $\tilde{P} = \Gamma(\mathcal{F})$. Using the Fourier decomposition of $e_{\mathbb{F}}(\Gamma)$ we can write

$$f(x) \stackrel{\text{def}}{=} e_{\mathbb{F}}(\tilde{P}(x)) = \sum_{i=1}^L c_i e_{\mathbb{F}}(\langle \alpha^{(i)}, \mathcal{F} \rangle(x)), \quad (8)$$

where $L = |\mathbb{F}|^{\dim(\mathcal{F})} = O_{\gamma, \beta}(1)$, $\alpha^{(i)} \in \mathbb{F}^{\dim(\mathcal{F})}$, and

$$\langle \alpha^{(i)}, \mathcal{F} \rangle(x) \stackrel{\text{def}}{=} \sum_{j=1}^m \alpha_j^{(i)} \cdot P_j(x).$$

Notice that the terms in (8), unlike Fourier characters, are not orthogonal. But since the factor is γ -uniform, [Lemma 4.5](#) ensures approximate orthogonality. Let $Q_i := \langle \alpha^{(i)}, \mathcal{F} \rangle$. Choose $\gamma(u) \leq \frac{\sigma}{|\mathbb{F}|^{2u}}$, so that $\gamma(\dim(\mathcal{F})) \leq \frac{\sigma}{L^2}$, where $\sigma := \frac{\varepsilon^{2k+1}}{4}$. It follows from the near orthogonality of the terms in (8) by [Lemma 4.5](#) that

$$|c_i - \langle f, e_{\mathbb{F}}(Q_i) \rangle| \leq \frac{\sigma}{L}, \quad (9)$$

and

$$\left| \|f\|_2^2 - \sum_{i=1}^L c_i^2 \right| \leq \sigma. \quad (10)$$

Claim 5.6 *There exists $\delta'(\varepsilon, |\mathbb{F}|) \in (0, 1]$ such that the following holds. Assume that f and \mathcal{F} are as above. Then there is $i \in [L]$, for which $\deg(Q_i) \leq k$ and $|\langle f, e_{\mathbb{F}}(Q_i) \rangle| \geq \delta'$.*

Proof: We will induct on the degree of \mathcal{F} . Assume for the base case that \mathcal{F} is of degree k , i.e. $d = k + 1$, thus applying the following Cauchy-Schwarz inequality

$$\varepsilon^{2k+1} \leq \|f\|_{U^{k+1}}^{2k+1} \leq \|f\|_2^2 \|f\|_{\infty}^{2k+1-2}, \quad (11)$$

and (10) imply that there exists $i \in [L]$ such that $c_i^2 \geq \frac{\varepsilon^{2k+1} - \sigma}{L} = \frac{3\varepsilon^{2k+1}}{4L}$, which combined with (9) implies that $|\langle f, e_{\mathbb{F}}(Q_i) \rangle| \geq \frac{\varepsilon^{2k+1}}{2L}$.

Now for the induction step, assume that $d > k + 1$. We will decompose (8) into two parts, first part consisting of the terms of degree $\leq k$ and the second part consisting of the terms of degree strictly higher than k . Namely, letting $S := \{i \in [L] : \deg(Q_i) \leq k\}$ we write $f = g + h$ where $g \stackrel{\text{def}}{=} \sum_{i \in S} c_i e_{\mathbb{F}}(Q_i)$ and $h \stackrel{\text{def}}{=} \sum_{i \in [L] \setminus S} c_i e_{\mathbb{F}}(Q_i)$. Notice that by the triangle inequality of Gowers norm, our choice of γ , and the fact that \mathcal{F} is γ -uniform

$$\|h\|_{U^{k+1}} \leq \sum_{i \in [L] \setminus S} |c_i| \cdot \|e_{\mathbb{F}}(Q_i)\|_{U^{k+1}} \leq L \cdot \frac{\varepsilon^{2k+1}}{4L^2} = \frac{\varepsilon^{2k+1}}{4L},$$

and thus

$$\|g\|_{U^{k+1}} \geq \frac{\varepsilon}{2}.$$

Now the claim follows by the base case. ■

Let $\delta'(\varepsilon, |\mathbb{F}|)$ be as in the above claim. We will use the following theorem of Goldreich and Levin [GL89] which gives an algorithm to find all the large Fourier coefficients of $e_{\mathbb{F}}(\Gamma)$.

Theorem 5.7 (Goldreich-Levin theorem [GL89]) *Let $\zeta, \rho \in (0, 1]$. There is a randomized algorithm, which given oracle access to a function $\Gamma : \mathbb{F}^m \rightarrow \mathbb{F}$, runs in time $O(m^2 \log m \cdot \text{poly}(\frac{1}{\zeta}, \log(\frac{1}{\rho})))$ and outputs a decomposition*

$$\Gamma = \sum_{i=1}^{\ell} b_i \cdot e_{\mathbb{F}}(\langle \eta_i, x \rangle) + \Gamma',$$

with the following guarantee:

- $\ell = O(\frac{1}{\zeta^2})$.
- $\mathbb{P}[\exists i : |b_i - \widehat{\Gamma}(\eta_i)| > \zeta/2] \leq \rho$.
- $\mathbb{P}[\forall \alpha \text{ such that } |\widehat{f}(\alpha)| \geq \zeta, \exists i \eta_i = \alpha] \geq 1 - \rho$.

We will use the above theorem with parameters $\zeta := \frac{\delta'}{2}$ and $\rho := \frac{\beta}{6}$, and use the randomized algorithm in Lemma 3.6 to provide answer to all its queries to Γ . Choose γ suitably small, so that with probability at least $1 - \frac{\beta}{6}$ our answer to all queries to Γ are correct. By Claim 5.6 there is $i \in [L]$ such that $\widehat{\Gamma}(\alpha) = c_i \geq \frac{3\delta'}{4}$. With probability $1 - \frac{\beta}{6}$ there is j such that $\eta_j = \alpha_i$ and with probability at least $1 - \frac{\beta}{6}$, $|b_j - c_i| \leq \frac{\zeta}{2} \leq \frac{\delta'}{4}$, and therefore $c_i \geq \frac{\delta'}{2}$.

By a union bound, adding up the probabilities of the errors, with probability at least $1 - \frac{5\beta}{6}$, we find Q_i such that

$$|\langle f, e_{\mathbb{F}}(Q_i) \rangle| \geq \frac{\delta'}{4}.$$

The following claim shows that, there is a constant shift of $e_{\mathbb{F}}(Q_i)$ that approximates f .

Claim 5.8 *Recall that $f(x) = e_{\mathbb{F}}(\tilde{P}(x)) = e_{\mathbb{F}}(t \cdot P(x))$, and we have*

$$|\langle e_{\mathbb{F}}(\tilde{P}), e_{\mathbb{F}}(Q_i) \rangle| = \left| \mathbb{E}_{x \in \mathbb{F}^n} e_{\mathbb{F}}(\tilde{P}(x) - Q_i(x)) \right| \geq \frac{\delta'}{4}.$$

There is a randomized algorithm that with probability $1 - \frac{\beta}{6}$ returns an $h \in \mathbb{F}$ for which

$$\mathbb{P}(\tilde{P}(x) = Q_i(x) + h) \geq \frac{1}{|\mathbb{F}|} + \frac{\delta'}{8|\mathbb{F}|^2}$$

Proof: Since $\tilde{P}(x) - Q_i(x)$ takes values in \mathbb{F} , there must be a choice of $r \in \mathbb{F}$ such that $\mathbb{P}(\tilde{P}(x) - Q_i(x) = r) \geq \frac{1}{|\mathbb{F}|} + \frac{\delta'}{4|\mathbb{F}|^2}$. Similar to the proof of [Lemma 2.1](#) defining $\mu_0(r) = \mathbb{P}(\tilde{P}(x) - Q_i(x) = r)$, we can find the estimate measure μ_{obs} by $K = O_{|\mathbb{F}|, \delta', \beta}(1)$ random queries to $\tilde{P}(x) - Q_i(x)$ such that

$$\mathbb{P}\left(\exists r \in \mathbb{F} : |\mu_0(r) - \mu_{\text{obs}}(r)| \geq \frac{\delta'}{8|\mathbb{F}|^2}\right) \leq \frac{\beta}{6}.$$

Let $h \in \mathbb{F}$ be such that $\mu_{\text{obs}}(h)$ is maximized. Letting $Q' := Q_i + h$ we have

- $\mathbb{P}(\tilde{P}(x) = Q'(x)) \geq \frac{1}{|\mathbb{F}|} + \frac{\delta'}{8|\mathbb{F}|^2}$
- $\deg(Q') = k$.

■

Since $\tilde{P} = t \cdot P$, the same also holds between P and $t^{-1}Q'$. The probability that all steps of the algorithm work correctly is bounded from below by $1 - \frac{5\beta}{6} - \frac{\beta}{6} = 1 - \beta$. ■

6 Low Characteristics

As mentioned in [Section 4](#), notions of regularity ([Definition 3.1](#)) and uniformity ([Definition 3.7](#)) are not sufficient to prove [Proposition 4.3](#) and [Proposition 4.4](#), respectively, without the assumption on $|\mathbb{F}|$ being larger than d . In this section we will use strongly unbiased factors ([Definition 3.14](#)) to handle the case of low characteristics.

Definition 3.14 (restated). Suppose that $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function. Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a polynomial factor of degree d over \mathbb{F}^n and let $\Delta : \mathcal{F} \rightarrow \mathbb{N}$ assign a natural number to each polynomial in the factor. We say that \mathcal{F} is strongly γ -unbiased with degree bound Δ if

1. For every $i \in [m]$, $1 \leq \Delta(P_i) \leq \deg(P_i)$.
2. For every $i \in [m]$ and $r > \Delta(P_i)$, there exists a function $\Gamma_{i,r}$ such that

$$P_i(x + y_{[r]}) = \Gamma_{i,r}(P_j(x + y_J) : j \in [m], J \subseteq [r], |J| \leq \Delta(P_j)).$$

3. For every $r \geq 0$ and collection of coefficients $\mathcal{C} = \{c_{i,I} \in \mathbb{F}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ not all of which are zero, we have

$$\text{bias}\left(\sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} P_i(x + y_I)\right) \leq \gamma(\dim(\mathcal{F})),$$

for random variables $x, y_1, \dots, y_r \in \mathbb{F}^n$.

We will say \mathcal{F} is Δ -bounded if it only satisfies (1) and (2). Moreover for every $r > 0$ let

$$B(r) \stackrel{\text{def}}{=} \sum_{1 \leq i \leq m} \sum_{0 \leq j \leq \Delta(P_i)} \binom{r}{j},$$

be the number of pairs (i, I) with $i \in [m]$, $I \subseteq [r]$, and $|I| \leq \Delta(P_i)$.

The following claim states that although in part (3) there is no upper bound on r , choosing a suitably faster decreasing function $\gamma' = \gamma^{2^d}$, one can assume that $r \leq d$.

Claim 6.1 Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a polynomial factor of degree d , and $\Delta : \mathcal{F} \rightarrow \mathbb{N}$ be a function such that \mathcal{F} is Δ -bounded. Suppose that $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function such that for every $r \in [d]$, $x \in \mathbb{F}^n$ and not all zero collection of coefficients $\{c_{i,I}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$

$$\text{bias} \left(\sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} \cdot P_i(x + y_I) \right) \leq \gamma(\dim(\mathcal{F}))^{2^d}, \quad (12)$$

where y_1, \dots, y_r are variables from \mathbb{F}^n . Then \mathcal{F} is strongly γ -unbiased.

Proof: Let $r > d$ be an integer, $x \in \mathbb{F}^n$ be fixed and $\{c_{i,I}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ be a set of coefficients, not all of which are zero. We will show that

$$\text{bias} \left(\sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} \cdot P_i(x + y_I) \right) \leq \gamma(\dim(\mathcal{F})).$$

Define $Q_{\mathcal{C}}(y_1, \dots, y_r) \stackrel{\text{def}}{=} \sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} \cdot P_i(x + y_I)$, and let $I \subseteq [r]$ be maximal such that $c_{i,I}$ is nonzero for some $i \in [m]$. Without loss of generality assume that $I = \{1, 2, \dots, |I|\}$. We will derive $Q_{\mathcal{C}}$ in directions $y_1, \dots, y_{|I|}$, namely

$$DQ_{\mathcal{C}, y_1, \dots, y_r}(h_1, \dots, h_{|I|}) \stackrel{\text{def}}{=} D_{h_1} \cdots D_{h_{|I|}} Q_{\mathcal{C}}(y_1, \dots, y_r),$$

where h_i is only supported on y_i coordinates. Notice that, since I was maximal, $DQ_{\mathcal{C}}$ does not depend on any y_i with $i \in [r] \setminus I$, namely, for every such $J \subseteq [r]$ with $J \not\subseteq I$, and every $i \in [m]$, $D_{h_1} \cdots D_{h_{|I|}} P_i(x + y_J) \equiv 0$. Moreover

$$\begin{aligned} D_{h_1} \cdots D_{h_{|I|}} Q_{\mathcal{C}}(y_1, \dots, y_r) &= \sum_{i \in [m], J \subseteq [r], |J| \leq \Delta(P_i)} c_{i,J} \cdot D_{h_1} \cdots D_{h_{|I|}} P_i(x + y_J) \\ &= \sum_{i \in [m]} c_{i,I} \cdot D_{h_1} \cdots D_{h_{|I|}} P_i(x + y_I) \\ &= \sum_{i \in [m], J \subseteq I} c_{i,I} \cdot (-1)^{|J|} \cdot P_i((x + y_I) + h_J). \end{aligned}$$

It follows by [Claim 4.7](#) for the random variable $x' := x + y_I$ and taking the bias over random variables $x', h_1, \dots, h_{|I|}$ that

$$\gamma(\dim(\mathcal{F}))^{2^d} \geq \text{bias} (DQ_{\mathcal{C}, y_1, \dots, y_r}(x', h_1, \dots, h_{|I|})) \geq \text{bias} (Q_{\mathcal{C}}(y_1, \dots, y_r))^{2^{|I|}}.$$

Now the claim follows by the fact that $|I| \leq d$. ■

The above claim suggests a way of refining a factor to a strongly uniform one: We will estimate the bias of every possible linear combination of the polynomials over all points $x + y_I$ where $I \subseteq [r]$ and $r \leq d$, and once we have detected a biased combination, refine using [Lemma 2.1](#). This is made possible by the fact that r and $\dim(\mathcal{F})$ are both bounded.

Lemma 6.2 (Algorithmic strongly γ -unbiased refinement) Suppose that $\sigma, \beta \in (0, 1]$ are parameters, and $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ is a decreasing function. There is a randomized algorithm that given a factor $\mathcal{F} = \{P_1, \dots, P_m\}$ of degree d , runs in $O_\gamma(n^d)$, with probability $1 - \beta$, returns a strongly γ -unbiased factor \mathcal{F}' with degree bound Δ such that \mathcal{F}' is σ -close to being a refinement of \mathcal{F} .

Proof: We will design a refinement process which has to stop in a finite number of steps which only depends on $\dim(\mathcal{F})$, σ , β , and γ . We will start with the initial factor being \mathcal{F} and we will set $\Delta(P_i) := \deg(P_i)$. Notice that this automatically satisfies the first two properties of [Definition 3.14](#), namely Δ -boundedness. This is because for every $r > \deg(P_i)$ we can use the derivative property

$$P_i(x + y_{[r]}) = \sum_{I \subseteq [r]} (-1)^{r-|I|} \cdot P_i(x + y_I).$$

to write $P_i(x + y_{[r]})$ as a function of $\{P_i(x + y_J)\}_{J \subseteq [r], |J| \leq d}$.

As explained in proof of [Lemma 4.1](#), at each step, we may assume without loss of generality that all polynomials in the factor are homogeneous. Moreover, we will assume that for every $r \leq d$, the set of polynomials $\{P_i(x + y_I)\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ is linearly independent, where $x, y_1, \dots, y_r \in \mathbb{F}^n$ are variables. This is because at each step we have access to explicit description of the polynomials in the factor, therefore we can compute each of the $|\mathbb{F}|^{B(r)}$ possible linear combinations, where $B(r) = \sum_{i \in [m]} \sum_{j \in [\Delta(P_i)]} \binom{r}{j}$, and check whether it is equal to zero or not. Suppose that for a set of coefficients $\mathcal{C} = \{c_{i,I} \in \mathbb{F}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ we have $Q_{\mathcal{C}} \stackrel{\text{def}}{=} \sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} P_i(x + y_I) \equiv 0$. Let P_i be a polynomial that appears in $Q_{\mathcal{C}}$ and let I be maximal such that $c_{i,I} \neq 0$. We can let $\Delta(P_i) := |I| - 1$, and remove P_i from the factor if $\Delta(P_i)$ becomes zero.

We will stop refining if \mathcal{F} is of degree 1. Notice that a linearly independent factor of degree 1 is not biased at all, and therefore strongly 0-unbiased. Assume that \mathcal{F} is not strongly γ -unbiased with respect to the current Δ , then there exists r and a set of coefficients $\mathcal{C} = \{c_{i,I} \in \mathbb{F}\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$ for which

$$\text{bias}(Q_{\mathcal{C}}) = \text{bias} \left(\sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} P_i(x + y_I) \right) > \gamma(\dim(\mathcal{F})).$$

To detect this, we will use the following algorithm: Set $K := |\mathbb{F}|^{B(r)}$. We can estimate the bias of each of the K possible linear combinations $Q_{\mathcal{C}}$ and check whether our estimate is greater than $\frac{3\gamma(\dim(\mathcal{F}))}{4}$. Letting $\beta' := \frac{\beta}{4K}$, we can distinguish $\text{bias} \geq \gamma(\dim(\mathcal{F}))$ from $\text{bias} \leq \frac{\gamma(\dim(\mathcal{F}))}{2}$ correctly with probability $1 - \beta'$ by computing the average of $e_{\mathbb{F}}(Q_{\mathcal{C}})$ on $O_{\dim(\mathcal{F})}(\frac{1}{\gamma(\dim(\mathcal{F}))^2 \beta})$ random sets of vectors x, y_1, \dots, y_r . Let \mathcal{C} be such that the estimated bias for $Q_{\mathcal{C}}$ was greater than $\frac{3\gamma(\dim(\mathcal{F}))}{4}$, and let $k = \deg(Q_{\mathcal{C}})$. By a union bound with probability $1 - \frac{\beta}{4}$, $\text{bias}(Q_{\mathcal{C}}) \geq \frac{\gamma(\dim(\mathcal{F}))}{2}$, and by [Lemma 2.1](#) we can find, with probability $1 - \frac{\beta}{4}$, a set of polynomials $Q_1, \dots, Q_s : (\mathbb{F}^n)^{r+1} \rightarrow \mathbb{F}$ of degree $k - 1$ such that

- $Q_{\mathcal{C}}$ is $\frac{\sigma}{2}$ -close to a function of Q_1, \dots, Q_s ,
- $s \leq \frac{16|\mathbb{F}|^5}{\gamma(\dim(\mathcal{F})) \cdot \sigma \cdot \beta}$.

Moreover we know from the proof of [Lemma 2.1](#) that for each $j \in [s]$ we have

$$\begin{aligned} Q_j(x, y_1, \dots, y_r) &= Q_{\mathcal{C}}(x + h_x, y_1 + h_1, \dots, y_r + h_r) - Q_{\mathcal{C}}(x, y_1, \dots, y_r) \\ &= \sum_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)} c_{i,I} \cdot D_{h_x + h_I} P_i(x + y_I). \end{aligned}$$

for some fixed vectors $h_x, h_1, \dots, h_r \in \mathbb{F}^n$. Let $R_{i,I}^{(j)} \stackrel{\text{def}}{=} D_{h_x + h_I} P_i$ so that $\deg(R_{i,I}^{(j)}) < \deg(P_i)$ and Q_j is measurable in $\{R_{i,I}^{(j)}(x + y_I)\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$.

Let P_i be a polynomial of maximum degree that appears in Q_C , and let I be maximal such that $c_{i,I} \neq 0$. We will add each $R_{i,I}^{(j)}$ to \mathcal{F} with $\Delta(R_{i,I}^{(j)}) = \deg(R_{i,I}^{(j)})$ and let $\Delta(P_i) := |I| - 1$ and discard P_i if $|I|$ becomes zero. Notice that the new factor is Δ -bounded because $P_i(x + y_I)$ can be written as a function of $\{Q_j(x, y_1, \dots, y_r)\}_{j \in [s]}$ and $\{P_j(x + y_J)\}_{J \subseteq I, |J| \leq \Delta(P_i)}$. Moreover each Q_j is measurable in $\{R_{i,I}^{(j)}(x + y_I)\}_{i \in [m], I \subseteq [r], |I| \leq \Delta(P_i)}$.

We can prove that the process stops after a constant number of steps by a strong induction on the dimension vector (M_1, \dots, M_d) of the factor \mathcal{F} , where M_i is the number of polynomials of degree i in the factor and d is the degree of the factor. The induction step follows from the fact that at every step we discard P_i or decrease the $\Delta(P_i)$ for some i and add polynomials of lower degree. Let \mathcal{F}' and $\Delta : \mathcal{F}' \rightarrow \mathbb{N}$ be the output of the algorithm. The claim follows by our choices for error distances and the probabilities. \blacksquare

6.1 From Approximation to Computation

Our goal is to prove the following analogue of [Lemma 3.12](#) for our notion of strongly unbiased factors.

Lemma 6.3 (Approximation by Strongly Unbiased Factor implies Computation) *For every $d \geq 0$ there exists a constant σ_d and a decreasing function $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ such that the following holds. Let $P : \mathbb{F}^n \rightarrow \mathbb{F}$ be a polynomial of degree d , $\mathcal{F} = \{P_1, \dots, P_m\}$ be a strongly γ -unbiased polynomial factor of degree $d - 1$ with degree bound Δ , and let $\Gamma : \mathbb{F}^m \rightarrow \mathbb{F}$ be a function such that P is σ_d -far from $\Gamma(\mathcal{F})$. Then P is in fact measurable in \mathcal{F} .*

Remark 6.4 (Exact refinement) *One can use the above lemma to modify the refinement process of [Lemma 6.2](#) to achieve exact refinement instead of approximate refinement. This can be done by using induction on the degree of the factor and at every round refining the new polynomials that are to be added to the factor to a strongly unbiased set of polynomials. Then [Lemma 6.3](#) ensures that the removed polynomial is measurable in the new set of polynomials and therefore at every step we have exact refinement of the original factor.*

In order to prove [Lemma 6.3](#), we will try to adapt the proof by [\[KL08\]](#) to our setting. To do this we will need some tools. First lemma which follows immediately from parts (1) and (2) of [Definition 3.14](#).

Lemma 6.5 ([KL08]) *Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a Δ -bounded factor. Let $x, x' \in \mathbb{F}^n$ be two fixed vectors for which $P_i(x) = P_i(x')$ for all $i \in [m]$, namely x and x' belong to the same atom of \mathcal{F} . Let $z_1, \dots, z_k \in \mathbb{F}^n$ be vectors for some $k \geq 1$, and let $Y_1, \dots, Y_k \in \mathbb{F}^n$ be k random variables. Then the following two events are equivalent:*

1. $A = [P_i(x + Y_I) = P_i(x' + z_I) \text{ for all } i \in [m] \text{ and } I \subseteq [k]]$
2. $B = [P_i(x + Y_I) = P_i(x' + z_I) \text{ for all } i \in [m] \text{ and } I \subseteq [k], 1 \leq |I| \leq \Delta(P_i)].$

The following equidistribution property of strongly unbiased factors will be a main building block of the proof of [Lemma 6.3](#).

Lemma 6.6 (Equidistribution of strongly unbiased factors) *Let $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$ be a decreasing function. Let $\mathcal{F} = \{P_1, \dots, P_m\}$ be a strongly γ -unbiased factor of degree d with degree bound Δ . Let $Y = (Y_1, \dots, Y_k) \in (\mathbb{F}^n)^k$ be random variables. For any non-empty $I \subseteq [k]$, let $x^I \in \mathbb{F}^n$ be fixed and $a^{(I)} = (a_1^{(I)}, \dots, a_k^{(I)}) \in \mathbb{F}^k$ be a vector such that*

- For every $i \in I$, $a_i^{(I)} \neq 0$,
- For every $i \notin I$, $a_i^{(I)} = 0$

Then the joint distribution of

$$\left(P_i \left(x^I + \sum_{j \in I} a_j^{(I)} Y_j \right) : i \in [m], I \subseteq [k], 1 \leq |I| \leq \Delta(P_i) \right)$$

is $\gamma(\dim(\mathcal{F}))^{1/2^d}$ -close to the uniform distribution on $\mathbb{F}^{B(k)}$.

Proof: Similar to the proof of [Lemma 4.9](#), through Fourier analysis it suffices to show that each nonzero linear combination of polynomials $P_i(x^I + \sum_{j \in I} a_j^{(I)} Y_j)$ has small bias. Suppose that $\mathcal{C} = \{c_{i,I} \in \mathbb{F}\}_{i \in [m], I \subseteq [k], 1 \leq |I| \leq \Delta(P_i)}$ is a collection of coefficients, not all of which are zero. Let

$$Q_{\mathcal{C}}(Y_1, \dots, Y_k) \stackrel{\text{def}}{=} \sum_{i \in [m], I \subseteq [k], |I| \leq \Delta(P_i)} c_{i,I} \cdot P_i(x^I + Y_I).$$

Let J be a set with maximal support such that there is a nonzero $c_{i,J}$ for some $i \in [m]$. Without loss of generality assume that $J = \{1, \dots, r\}$, for $r = |J|$. We will derive $Q_{\mathcal{C}}$ in directions of Y_1, \dots, Y_r . Namely,

$$DQ(Y_1, \dots, Y_k, h_1, \dots, h_r) \stackrel{\text{def}}{=} D_{h_1} \cdots D_{h_r} Q_{\mathcal{C}}(Y_1, \dots, Y_k),$$

where h_i is only supported on Y_i coordinates. Then we have

$$\begin{aligned} DQ(Y_1, \dots, Y_k, h_1, \dots, h_r) &= \sum_{i \in [m], I \subseteq [k], |I| \leq \Delta(P_i)} c_{i,I} D_{h_1} \cdots D_{h_r} (P_i(x^I + \sum_{j \in I} a_j^{(I)} Y_j)) \\ &= \sum_{i \in [m]: \Delta(P_i) \geq |J|} c_{i,J} \cdot (D_{z_1} \cdots D_{z_r} P_i)(x^J + \sum_{j \in J} a_j^{(J)} Y_j) \\ &= \sum_{i \in [m]: \Delta(P_i) \geq |J|} c_{i,J} \cdot \sum_{I \subseteq J} (-1)^{|I|} P_i(x^J + \sum_{j \in J} a_j^{(J)} Y_j + z_I), \end{aligned}$$

where z_i is the projection of h_i on Y_I coordinates. Now by the change of variable $X := x^J + \sum_{j \in J} a_j^{(J)} Y_j$, we have

$$DQ(Y_1, \dots, Y_k, h_1, \dots, h_r) = \sum_{i \in [m], I \subseteq J, |I| \leq \Delta(P_i)} (-1)^{|I|} c_{i,J} \cdot P_i(X + z_I).$$

Now \mathcal{F} being strongly γ -unbiased with degree bound Δ and [Claim 4.7](#) imply that

$$\text{bias}(Q_{\mathcal{C}})^{2^{|J|}} \leq \text{bias}(DQ) \leq \gamma(\dim(\mathcal{F})).$$

■

Following is an immediate corollary of [Lemma 6.5](#) and [Lemma 6.6](#).

Corollary 6.7 *Suppose that $\mathcal{F} = \{P_1, \dots, P_m\}$ is a strongly γ^{2^d} -unbiased factor of degree d with degree bound Δ , for some decreasing function $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$. Let $x, x' \in \mathbb{F}^n$ be two fixed vectors such that $P_i(x) = P_i(x')$ for all $i \in [m]$. Let $z_1, \dots, z_k \in \mathbb{F}^n$ be values for some $k \geq 1$, and let $Y_1, \dots, Y_k \in \mathbb{F}^n$ be k random variables. Then*

$$\mathbb{P} \left[P_i(x + Y_I) = P_i(x' + z_I) : \forall i \in [m], \forall I \subseteq [k] \right] = (1 \pm \gamma(\dim(\mathcal{F}))) \cdot |\mathbb{F}|^{-B(k)}.$$

The next lemma from [\[KL08\]](#) is the last technical tool we will need for the proof of [Lemma 6.3](#). This lemma can be adapted to our setting by [Corollary 6.7](#) and [Lemma 6.6](#).

Lemma 6.8 (Analogue of [\[KL08\] Lemma 7](#)) *Suppose that \mathcal{F} is a strongly γ^{2^d} -unbiased factor of degree d with degree bound Δ , for a decreasing function $\gamma : \mathbb{N} \rightarrow \mathbb{R}^+$. Let A be an atom of \mathcal{F} and $x \in \mathbb{F}^n$ be a point in A . Then:*

1. *Let Y_1, \dots, Y_{d+1} be random variables in \mathbb{F}^n . Then*

$$\mathbb{P} [x + Y_I \in A, \forall I \subseteq [d+1]] = (1 \pm \gamma(\dim(\mathcal{F}))) \cdot |\mathbb{F}|^{-B(d+1)}$$

2. *Let $Y_1, \dots, Y_{d+1}, Z_1, \dots, Z_{d+1}$ be random variables in \mathbb{F}^n . For any non-empty $I_0 \in [d+1]$*

$$\mathbb{P} \left[x + Y_I, x + Z_I \in A, \forall I \subseteq [d+1] \middle| Y_{I_0} = Z_{I_0} \right] \leq (1 + \gamma(\dim(\mathcal{F}))) |\mathbb{F}|^m \left(|\mathbb{F}|^{-B(d+1)} \right)^2.$$

Proof: Part (1) follows immediately from [Corollary 6.7](#). Proof of part (2) almost exactly mimics the proof of [Lemma 7](#) in [\[KL08\]](#). Their proof (which is a bit technical) only requires equidistribution properties proved in [Lemma 6.6](#), and thus follows directly from our notion of regularity. We omit the details. ■

Proof of [Lemma 6.3](#): The proof idea is similar to the proof of [Proposition 4.4](#). Let $\tilde{P} \stackrel{\text{def}}{=} \Gamma(\mathcal{F})$. Let X be the set of points in \mathbb{F}^n for which $P(x) = \tilde{P}(x)$. By assumption we know that $|X| \geq (1 - \sigma_d) |\mathbb{F}|^n$. We will use $(d+1)$ -dimensional parallelepipeds to prove that $P(x) = \tilde{P}(x)$ within the whole of most of the atoms of \mathcal{F} , and then to prove that P should be constant on each remaining atom as well.

By [Lemma 3.5](#) we have that for $1 - O(\sqrt{\sigma_d})$ of the atoms A of \mathcal{F} we have $\mathbb{P}_{x \in A} (P(x) = \tilde{P}(x)) \geq 1 - O(\sqrt{\sigma_d})$; we will say that these atoms are *almost good*. We will prove that almost good atoms are in fact, totally good.

Suppose that $A = \mathcal{F}^{-1}(t)$ is an almost good atom, where $t = (t_{i,j})_{1 \leq i \leq d, 1 \leq j \leq M_i}$ is a vector in $\mathbb{F}^{\dim(\mathcal{F})}$. Let $A' := A \cap X$ be the set of points in A for which $P(x) = \tilde{P}(x)$. We will prove that $A' = A$, namely P is equal to \tilde{P} on A .

Let $B = A \setminus A'$ be the set of bad points in the atom A . Choosing $\sigma_d \leq 2^{-4(d+1)}$, since A is a good atom then $|B| \leq 2^{-2(d+1)} |A|$. Assume that B is non-empty. Let Y_1, \dots, Y_{d+1} be random variables in \mathbb{F}^n . Choosing $x \in B$, by [Lemma 6.8](#) part (1),

$$p_A \stackrel{\text{def}}{=} \mathbb{P} [x + Y_I \in A, \forall I \subseteq [d+1]] \geq (1 - \gamma(\dim(\mathcal{F}))) \cdot |\mathbb{F}|^{-B(d+1)}.$$

We will upper bound the event that while all $X + Y_I$ are in A , $X + Y_J \in B$ for some J . To do this, we will apply Cauchy-Schwarz inequality to reduce the problem to counting pairs of hypercubes. Fix a non-empty set $I_0 \subseteq [d + 1]$, and let

$$\begin{aligned} p_B &\stackrel{\text{def}}{=} \mathbb{P} [x + Y_I \in A, \forall I \subseteq [d + 1] \text{ and } x + Y_{I_0} \in B] \\ &= \sum_{x_0 \in B} \mathbb{P} [x + Y_I \in A, \forall I \subseteq [d + 1] \text{ and } x + Y_{I_0} = x_0]. \end{aligned}$$

Let Z_1, \dots, Z_{d+1} be random variables from \mathbb{F}^n we have

$$\begin{aligned} p_B^2 &= \left(\sum_{x_0 \in B} \mathbb{P} [x + Y_I \in A, \forall I \subseteq [d + 1] \text{ and } x + Y_{I_0} = x_0] \right)^2 \\ &\leq |B| \sum_{x_0 \in B} \mathbb{P} [x + Y_I \in A, \forall I \subseteq [d + 1] \text{ and } x + Y_{I_0} = x_0]^2 \\ &= |B| \mathbb{P} \left[x + Y_I, x + Z_I \in A, \forall I \subseteq [d + 1] \text{ and } x + Y_{I_0} = x + Z_{I_0} \right] \\ &= |B| |\mathbb{F}|^{-n} \mathbb{P} \left[x + Y_I, x + Z_I \in A, \forall I \subseteq [d + 1] \middle| x + Y_{I_0} = x + Z_{I_0} \right] \\ &\leq |B| |\mathbb{F}|^{m-n} p_A^2 (1 + \gamma(\dim(\mathcal{F}))), \end{aligned}$$

where the first inequality is the Cauchy Schwarz inequality and the last inequality follows from [Lemma 6.8](#), part (2). By [Lemma 3.5](#), $|A| = (1 \pm \gamma(\dim(\mathcal{F}))) |\mathbb{F}|^{\dim(\mathcal{F})-n}$, thus

$$p_B^2 \leq \frac{|B|}{|A|} p_A^2 (1 \pm 2\gamma(\dim(\mathcal{F}))) \leq 2^{-2(d+1)} p_R^2,$$

and thus $\frac{p_B}{p_A} \leq 2^{-(d+1)} (1 \pm 2\gamma(\dim(\mathcal{F})))$. Now by a union bound over all non-empty $I_0 \subseteq [d + 1]$ the probability that there is some I_0 for which $x + Y_{I_0} \in B$ is strictly less than 1 for a small enough γ . Therefore, there exists $y_1, \dots, y_{d+1} \in \mathbb{F}^n$ for which $x + y_I \in A \setminus B$, for every non-empty $I \subseteq [d + 1]$. This implies $x \in B$, since $\deg(P) = d$ and therefore

$$P(x) = \sum_{I \subseteq [d+1], |I| \neq 0} (-1)^{|I|+1} P(x + y_I) = \sum_{I \subseteq [d+1], |I| \neq 0} (-1)^{|I|+1} \tilde{P}(x + y_I) = \tilde{P}(x).$$

This proves that every almost good atom is in fact totally good. It remains to prove that P is constant on each of the remaining $O(\sqrt{\sigma_d})$ of the atoms. Now the proof follows from [Claim 4.11](#). \blacksquare

6.2 Applications

6.2.1 Computing a biased Polynomial

Having [Lemma 6.2](#) and [Lemma 6.3](#) in hand we immediately have the following analogue of [Theorem 3.13](#), which states that if a polynomial is biased then we can find a factor that computes it.

Theorem 6.9 (Computing a biased polynomial) *Let $\beta \in (0, 1]$ be an error parameter. There is a randomized algorithm that given query access to a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree d such that $\text{bias}(P) \geq \delta > 0$, runs in $O_{\delta, \beta}(n^d)$ and with probability $1 - \beta$, returns a polynomial factor $\mathcal{F} = \{P_1, \dots, P_{c(d, \delta)}\}$ of degree $d - 1$ such that P is measurable in \mathcal{F} .*

Proof: Let σ_d and γ_d be as in [Lemma 6.3](#). By [Lemma 2.1](#) since $\text{bias}(P) \geq \delta$, in time $O_{\sigma_d, \beta}(n^d)$, with probability $1 - \frac{\beta}{2}$ we can find a polynomial factor \mathcal{F} of degree $d - 1$ such that

- P is $\frac{\sigma_d}{2}$ -close to a function of \mathcal{F}
- $\dim(\mathcal{F}) \leq \frac{4|\mathbb{F}|^5}{\delta^2 \sigma_d \beta}$

By [Lemma 6.2](#), with probability $1 - \frac{\beta}{2}$, we can find a strongly γ_d -unbiased factor \mathcal{F}' with degree bound Δ such that \mathcal{F}' is $\frac{\sigma_d}{2}$ -close to being a refinement of \mathcal{F} .

Thus with probability greater than $1 - \beta$, P is σ_d -close to a function of the strongly γ_d -unbiased factor \mathcal{F}' , and it follows from [Lemma 6.3](#) that P is measurable in \mathcal{F}' . ■

6.2.2 Worst Case to Average Case Reduction

Here we will show how [Theorem 6.9](#) implies an algorithmic version of worst case to average case reduction from [\[KL08\]](#). To present the result, we first have to define what it means for a factor to approximate a polynomial.

Definition 6.10 (δ -approximation) *We say that a function $f : \mathbb{F}^n \rightarrow \mathbb{F}$ δ -approximates a polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ if*

$$\left| \mathbb{E}_{x \in \mathbb{F}^n} [e_{\mathbb{F}}(P(x) - f(x))] \right| \geq \delta.$$

Kaufman and Lovett use [Lemma 3.12](#) to show the following reduction.

Theorem 6.11 (Theorem 3 of [\[KL08\]](#)) *Let $P(x)$ be a polynomial of degree k , g_1, \dots, g_c polynomials of degree d , and $\Lambda : \mathbb{F}^c \rightarrow \mathbb{F}$ a function such that composition $\Lambda(g_1(x), \dots, g_c(x))$ δ -approximates P . Then there exist c' polynomials $h_1, \dots, h_{c'}$ and a function $\Gamma : \mathbb{F}^{c'} \rightarrow \mathbb{F}$ such that*

$$P(x) = \Gamma(h_1(x), \dots, h_{c'}(x)).$$

Moreover, $c' = c'(d, c, k)$ and each h_i is of the form $p(x + a) - p(x)$ or $g_j(x + a)$, where $a \in \mathbb{F}^n$. In particular, if $k \leq d - 1$ then $\deg(h_i) \leq k - 1$ also.

Here we will design a randomized algorithm that given g_1, \dots, g_c can compute a set of $h_1, \dots, h_{c'}$ efficiently.

Theorem 6.12 (Worst-case to average case reduction) *Let $\delta, \beta \in (0, 1]$ be parameters. There is a randomized algorithm that takes as input*

- A polynomial $P : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree d
- A polynomial factor $\mathcal{F} = \{P_1, \dots, P_m\}$ of degree $d - 1$
- A function Λ such that $\Lambda(\mathcal{F})$ δ -approximates P

and with probability at least $1 - \beta$, returns a polynomial factor $\mathcal{F}' = \{R_1, \dots, R_{m'}\}$ and a function $\Gamma : \mathbb{F}^{\dim(\mathcal{F}')} \rightarrow \mathbb{F}$ such that

$$P(x) = \Gamma(R_1(x), \dots, R_{m'}(x)),$$

moreover $c' = O_{\dim(\mathcal{F}), \beta, \delta, d}(1)$.

Proof: Looking at the Fourier decomposition of $e_{\mathbb{F}}(\Lambda)(y_1, \dots, y_m)$, since $\Lambda(P_1, \dots, P_m)$ δ -approximates P , there must exist $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{F}^c$ such that $Q_\alpha(x) = \sum_{i \in [m]} \alpha_i \cdot P_i(x)$, δ' -approximates P , where $\delta' \geq \frac{\delta}{|\mathbb{F}|^m}$. We will estimate $|\text{bias}(P - Q_\alpha)|$ for every $\alpha \in \mathbb{F}^m$. For each α , we can distinguish $\text{bias}(P - Q_\alpha) \leq \frac{\delta'}{2}$ from $\text{bias}(P - Q_\alpha) \geq \delta'$, with probability $1 - \frac{\beta}{3|\mathbb{F}|^m}$, by evaluating $e_{\mathbb{F}}(P(x) - Q_\alpha(x))$ on $C = O_{\dim(\mathcal{F})} \left(\frac{1}{\delta'^2} \log\left(\frac{1}{\beta}\right) \right)$ random inputs. Let $\alpha^* \in \mathbb{F}^m$ be such that our estimate for $\text{bias}(P - Q_\alpha)$ is greater than $\frac{3\delta'}{4}$.

With probability at least $1 - \frac{\beta}{3|\mathbb{F}|^m}$ we will find such α^* , and by a union bound with probability at least $1 - \frac{\beta}{3}$, $\text{bias}(P - Q_{\alpha^*}) \geq \frac{\delta'}{2}$. Now applying [Theorem 6.9](#) to $P - Q_{\alpha^*}$ with parameters $\frac{\beta}{3}$ and $\frac{\delta'}{2}$, we find a polynomial factor $\mathcal{F}' = \{R_1, \dots, R_{\tilde{m}}\}$ of degree $d - 1$, such that with probability $1 - \frac{\beta}{3}$, $P - Q_{\alpha^*}$ is measurable in \mathcal{F}' . Namely there exists Γ' such that $P - Q_{\alpha^*}(x) = \Gamma'(\mathcal{F}'(x))$ and therefore

$$P(x) = \Gamma'(R_1(x), \dots, R_{\tilde{m}}(x)) + Q_{\alpha^*}(x).$$

■

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