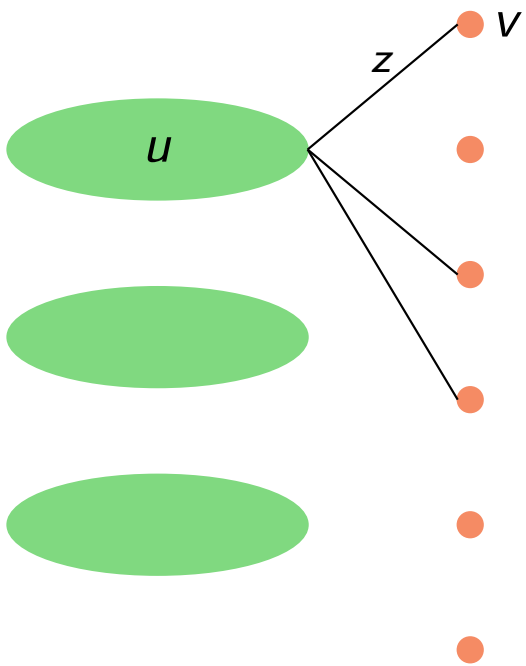
Towards an Optimal Query Efficient PCP?

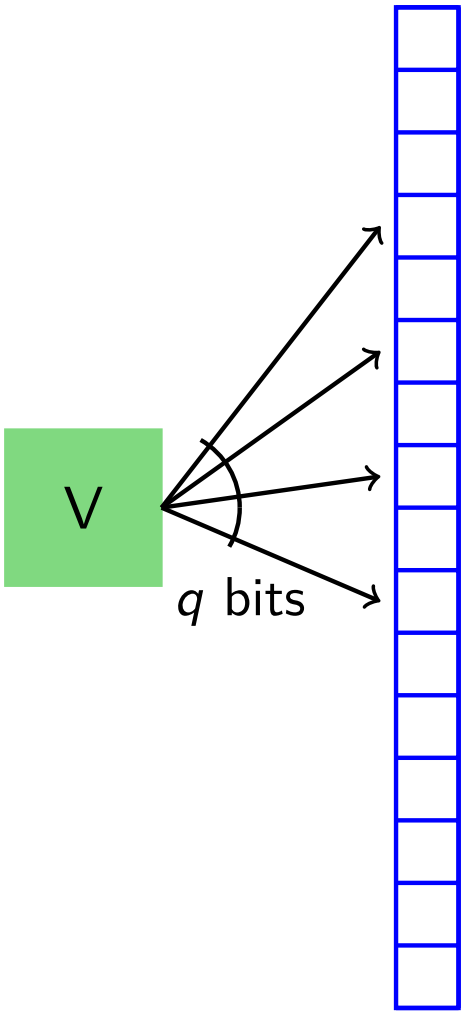


Subhash Khot
NYU and U. Chicago

Muli Safra
Tel Aviv University

Madhur Tulsiani
TTI Chicago

Probabilistically Checkable Proofs (PCPs)



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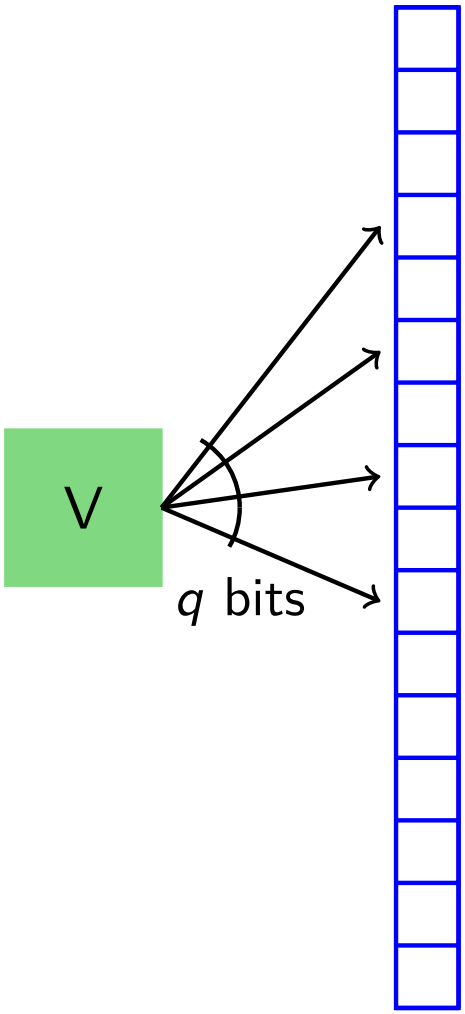

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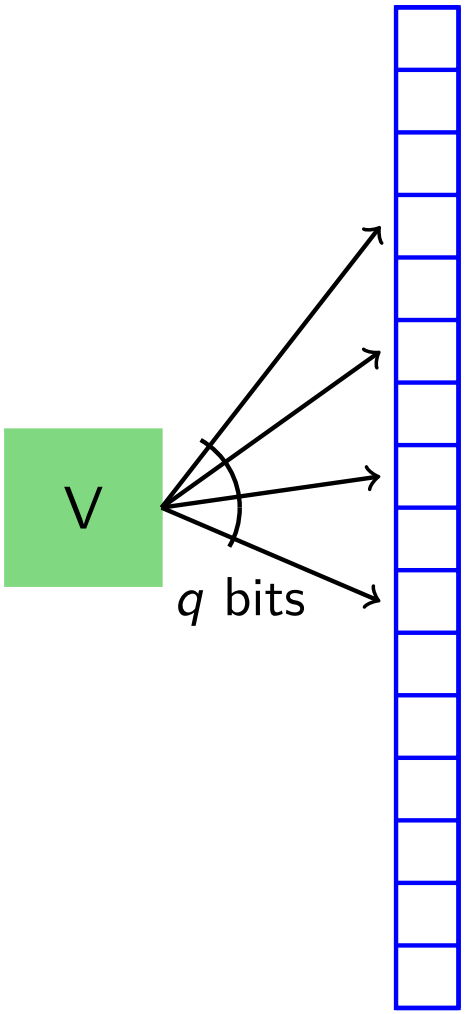

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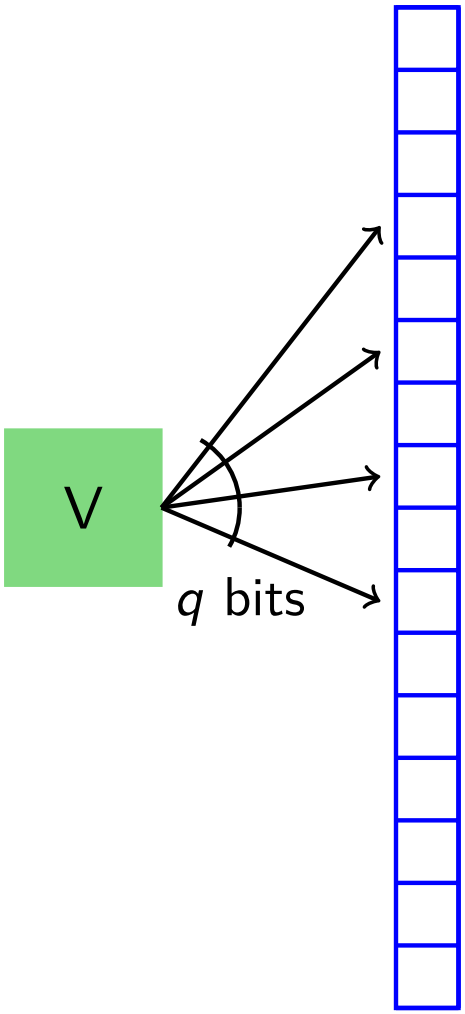
- Interested in tradeoff of q vs s

PCPs $\equiv$ Constraint Satisfaction



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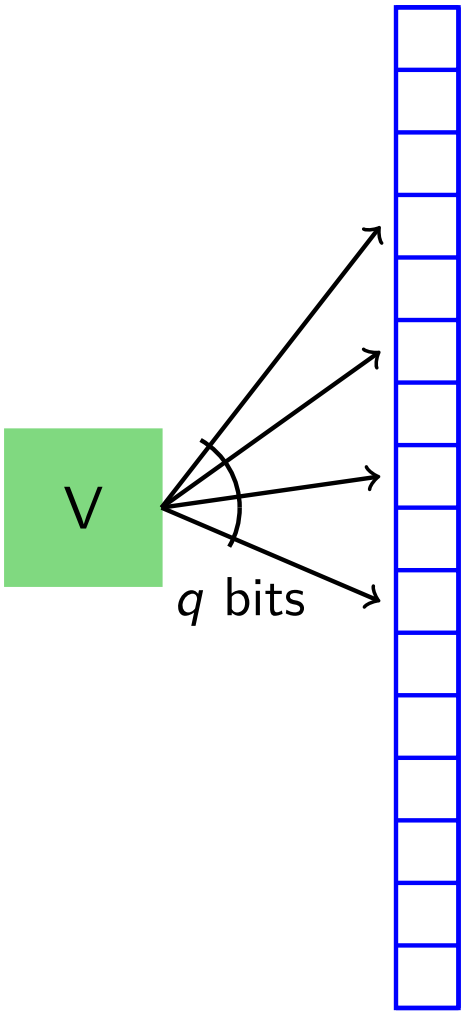

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  Verifier queries $\equiv$ constraints on q-tuples

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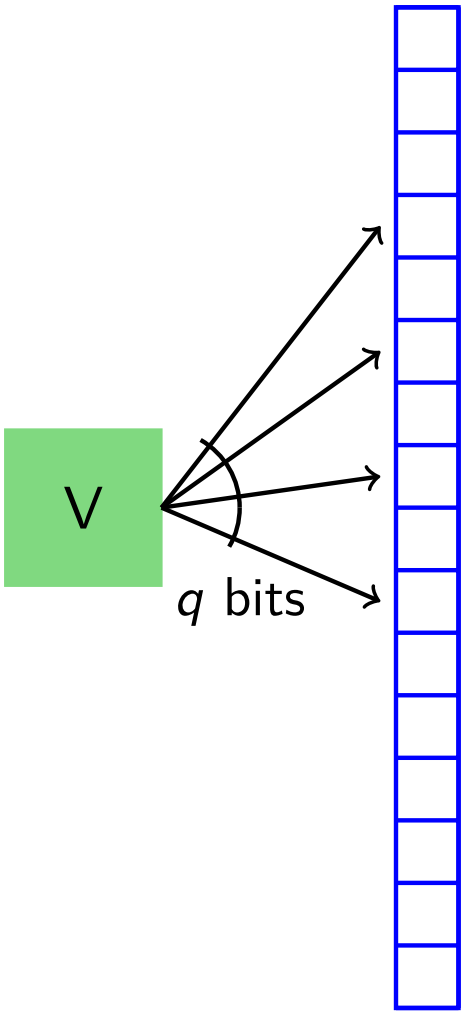

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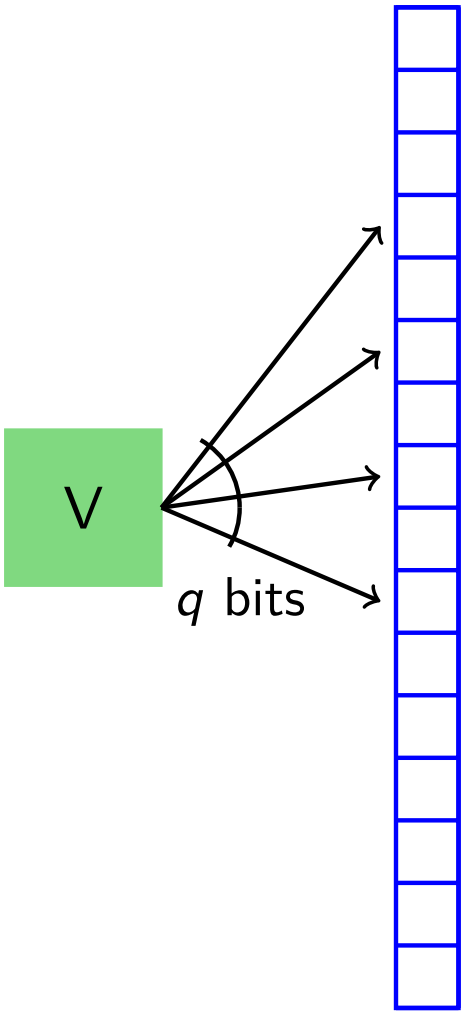

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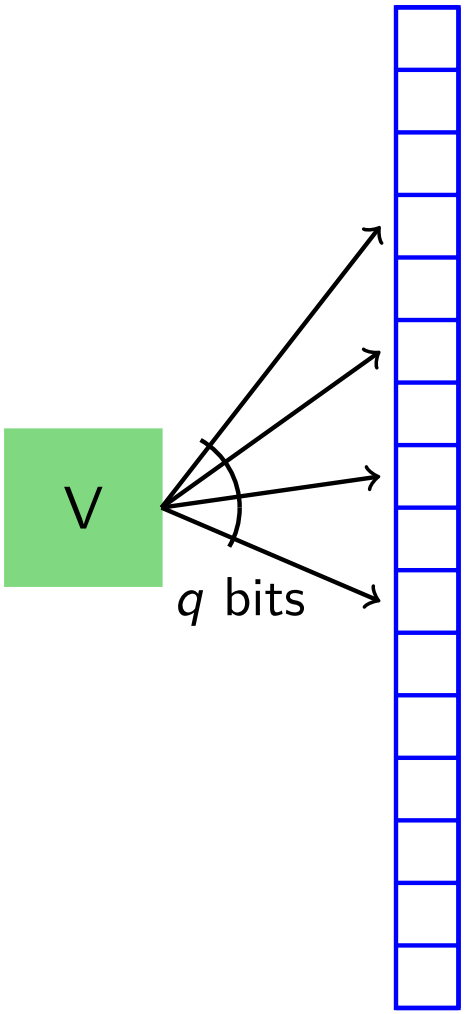

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- Gives hardness of approximating within factor $1/s$.
- Want to know best value of s using any kind of constraints on q variables.

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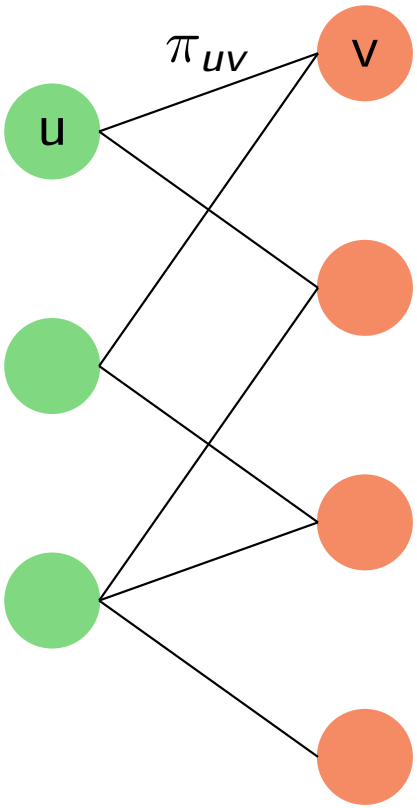
- Interesting because of new outer PCP.

Outer PCPs: Two-Prover Games



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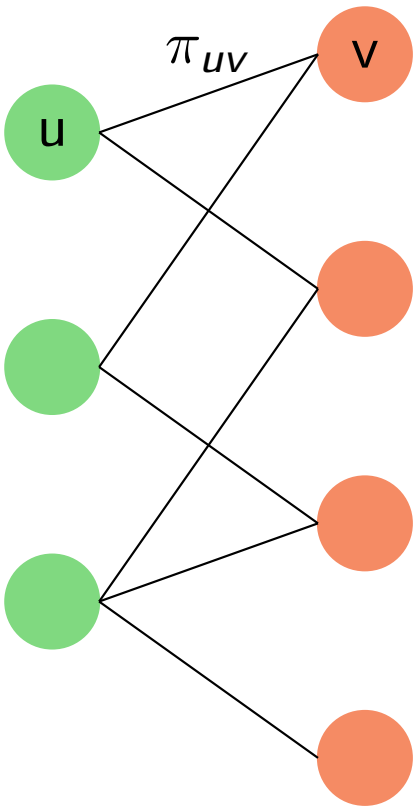

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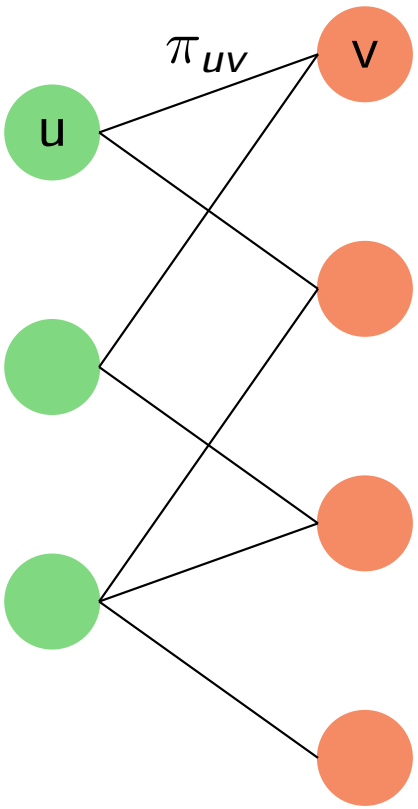

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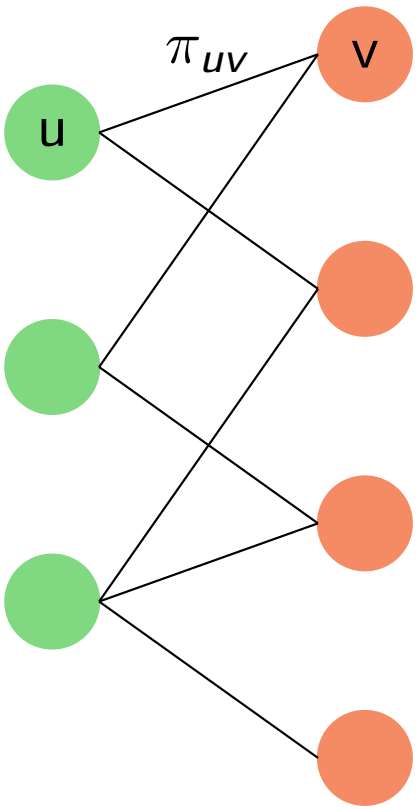

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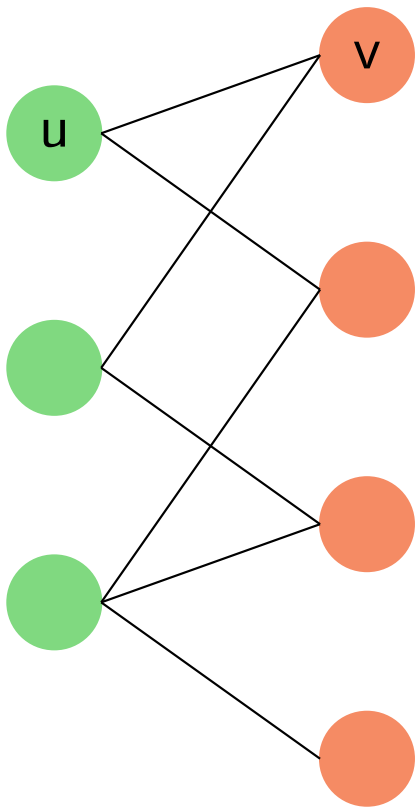
- Unique game when $\Sigma_1 = \Sigma_2$ and π_{uv}s are permutations.

Designing a PCP



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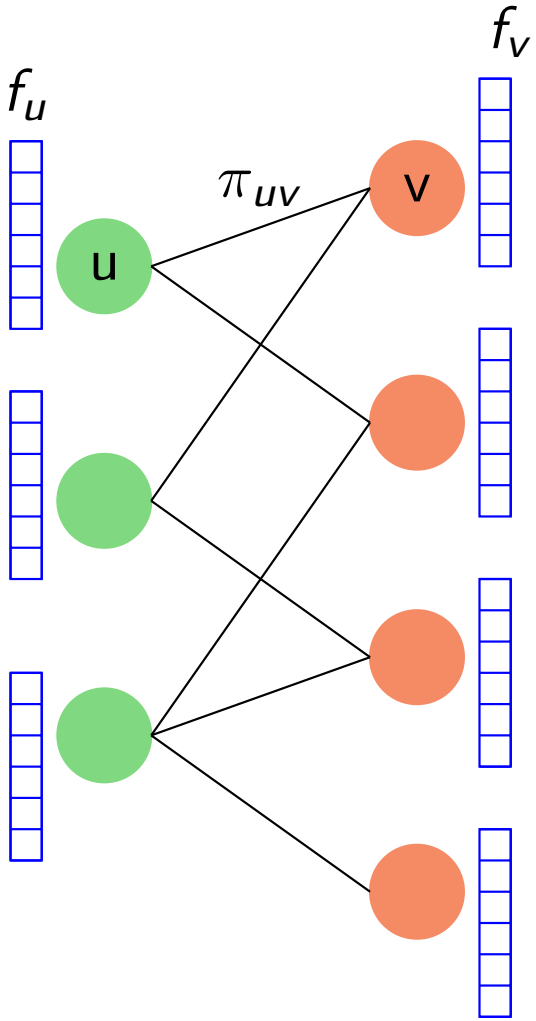

- Provers required to provide encodings of labels.

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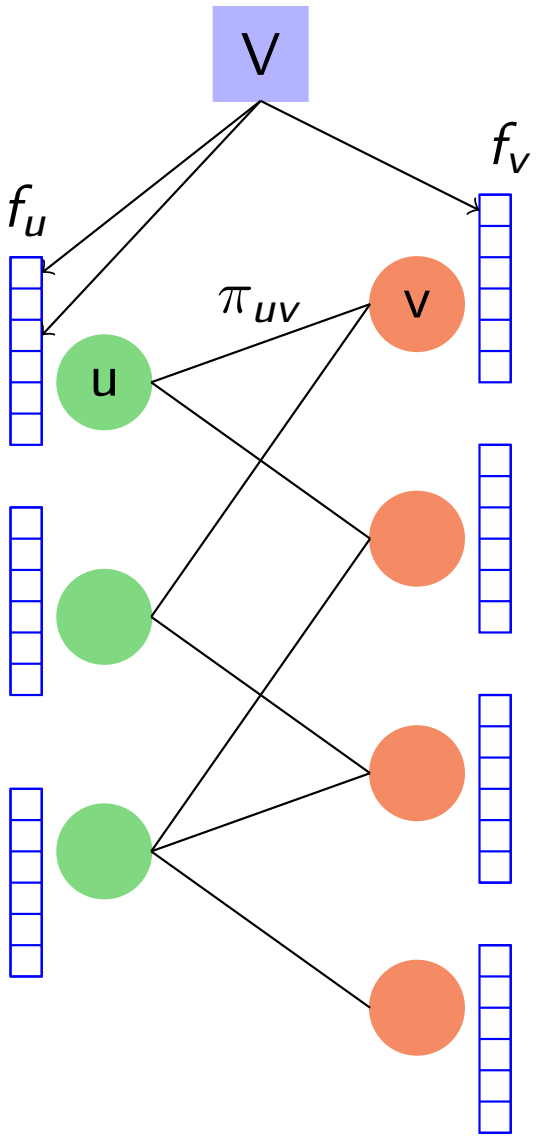

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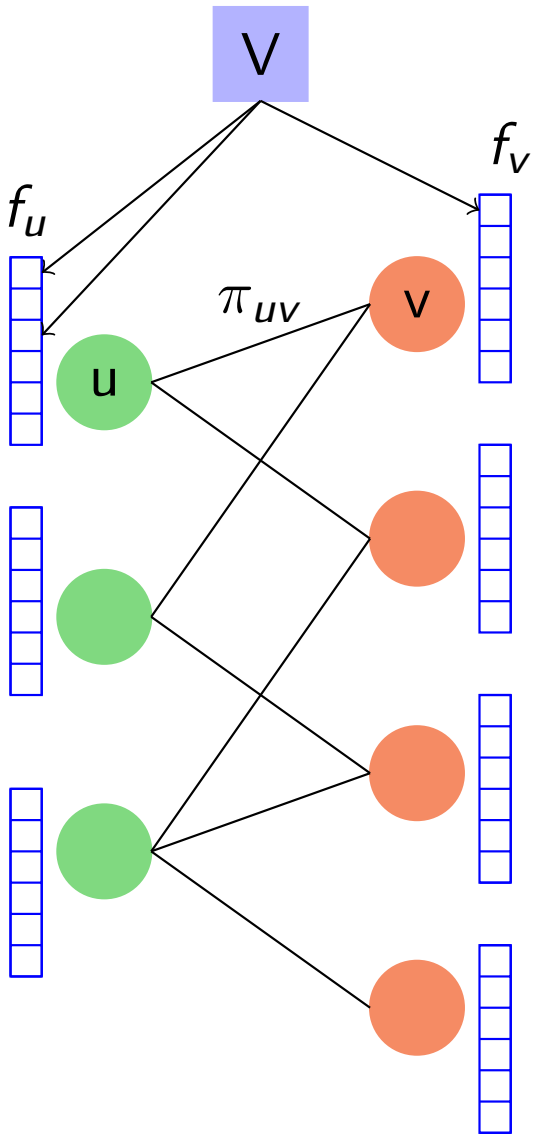

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- Test for valid encodings (inner PCP) governs the type of constraints. e.g.

  $$f(x) + f(y) = f(x + y)$$

  in [Håstad97]

The inner PCP for [ST06]

- For $f : \{0,1\}^R \to \{0,1\}$, test as follows:
  - Pick $x_1, \ldots, x_k \in \{0,1\}^R$ at random.
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- We give such an outer PCP.

The case of $k = 3$

- For all x_1, x_2, x_3, we have a constraint checking all of the following

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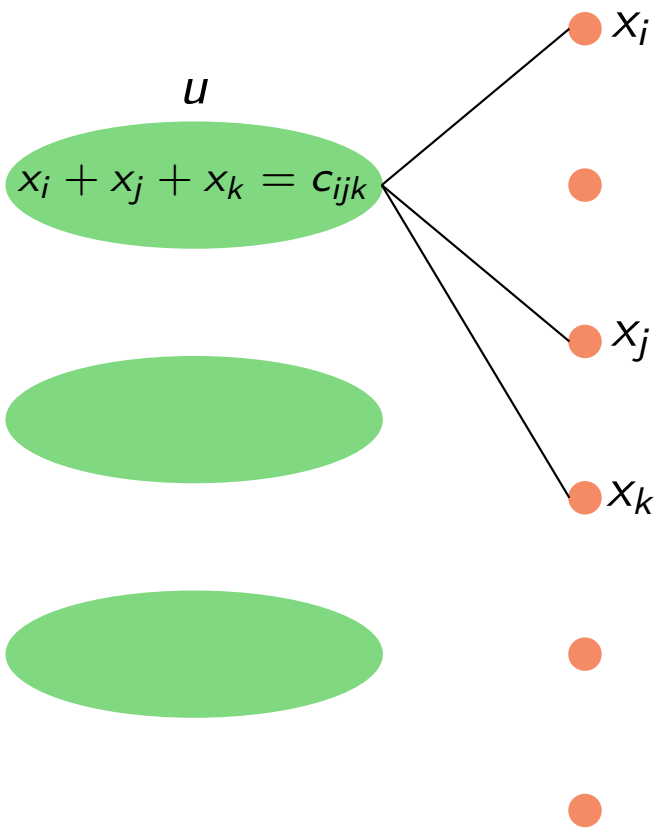
- Quadratic functions provide (somewhat) good encodings. Outer PCP needs to be robust against quadratic encodings.

Robustness against linear encodings (using [Håstad97])



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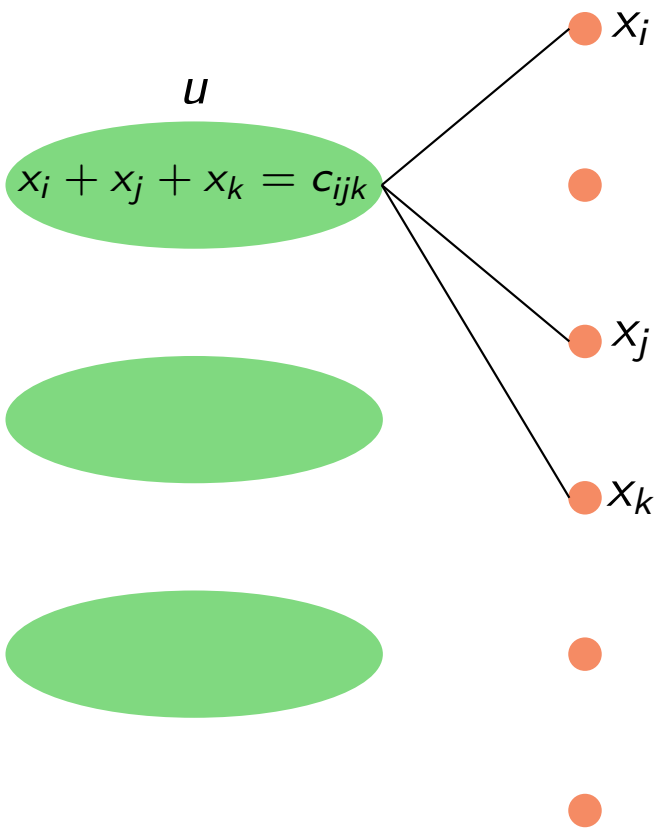

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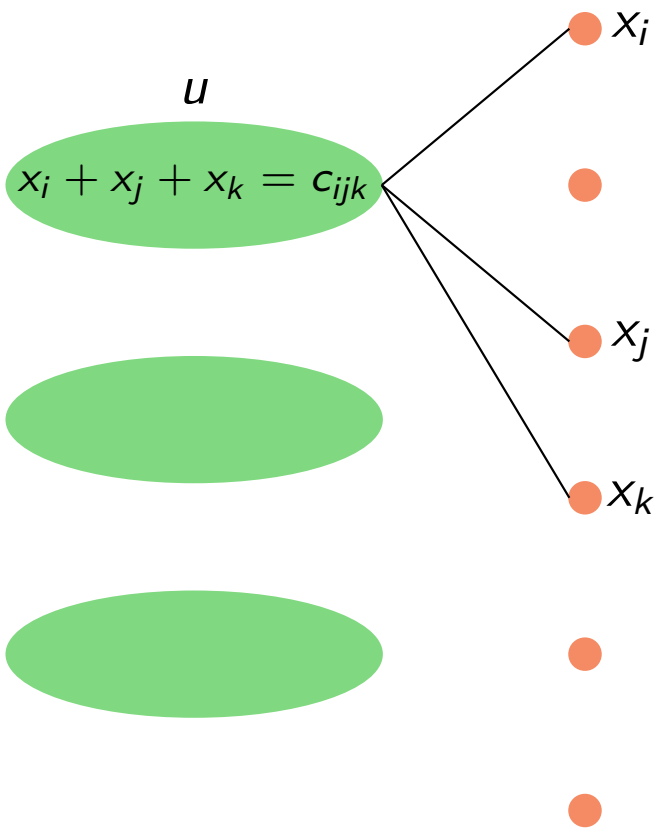

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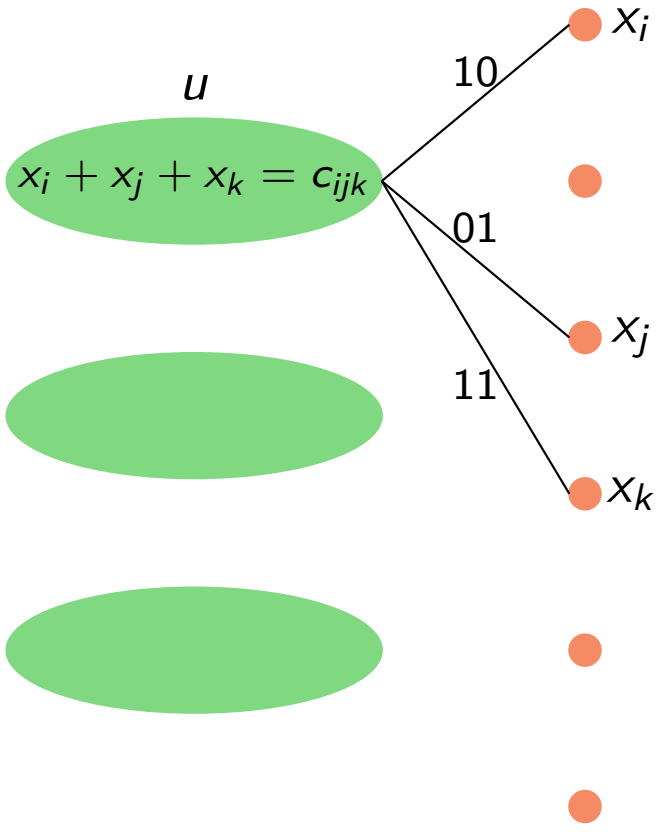

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- Constraints for different inputs to f_u.

$$\begin{aligned} f_u(1,0) &= L(x_i) \\ f_u(0,1) &= L(x_j) \\ f_u(1,1) &= L(x_k) + c_{ijk} \end{aligned}$$

(must have $L(x_i) + L(x_i) = L(x_k) + c_{ijk}$).

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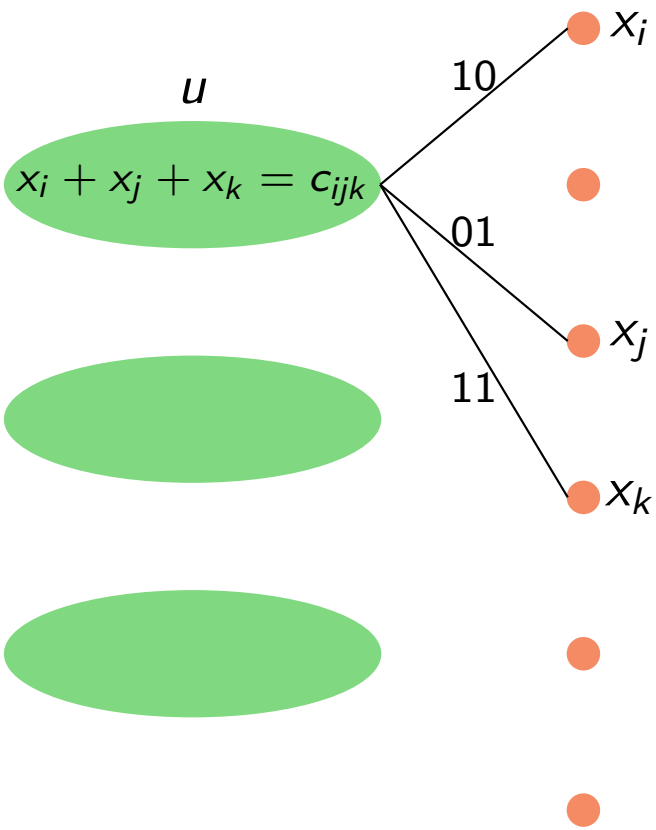

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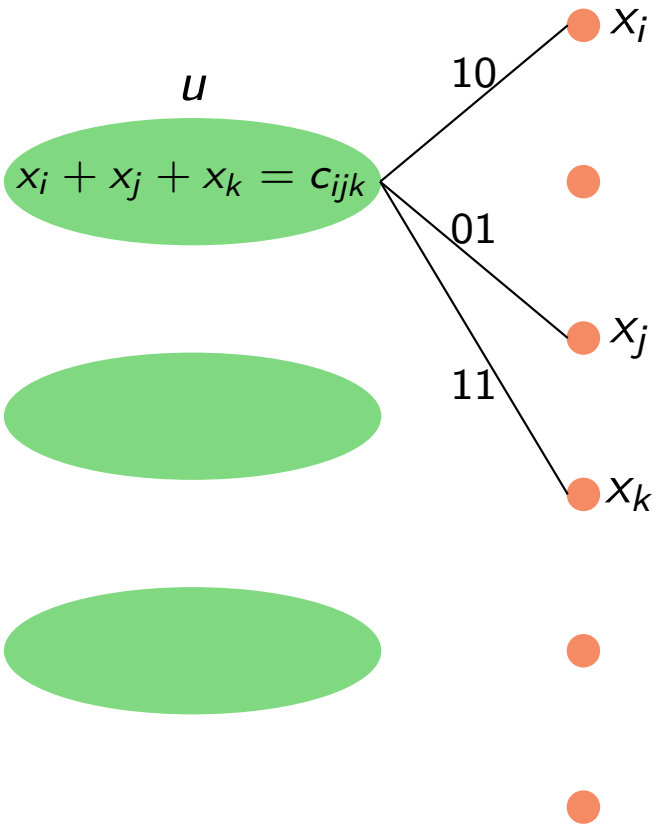

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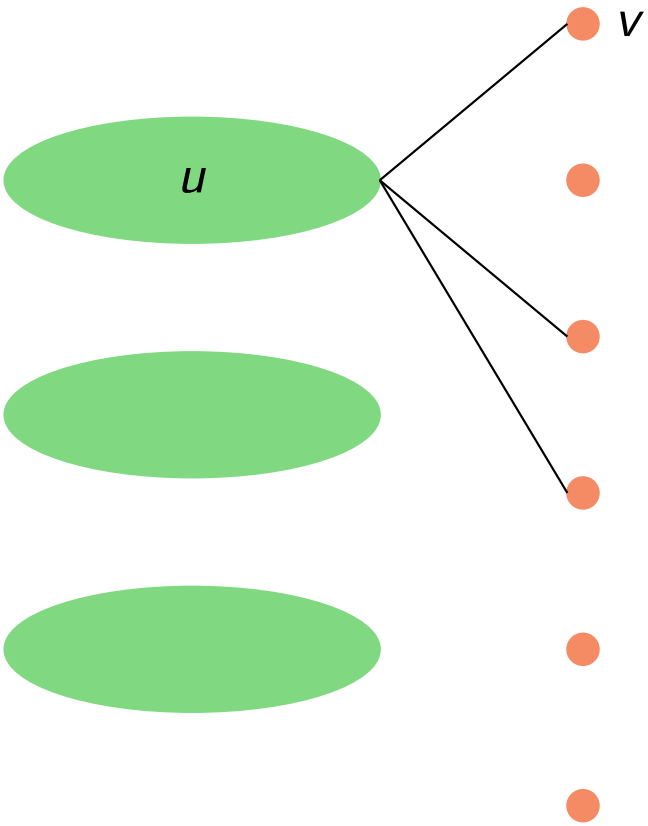
  - Almost all XOR constraints can be satisfied.
  - At most 1/2 of XOR constraints can be satisfied.
- Gives hardness of distinguishing between the following cases:
  - $\exists$ linear functions $\{f_u\}$ satisfying ≈ 1 fraction of constraints.
  - $\forall$ sets of linear functions $\{f_u\}$, at most 5/6 fraction of constraints are satisfied.

Robustness against degree-d polynomials



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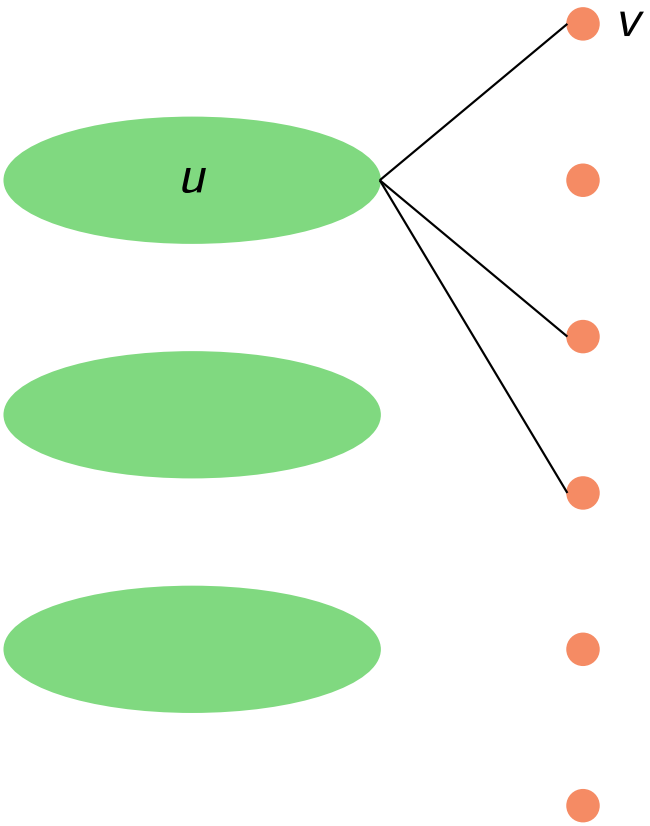

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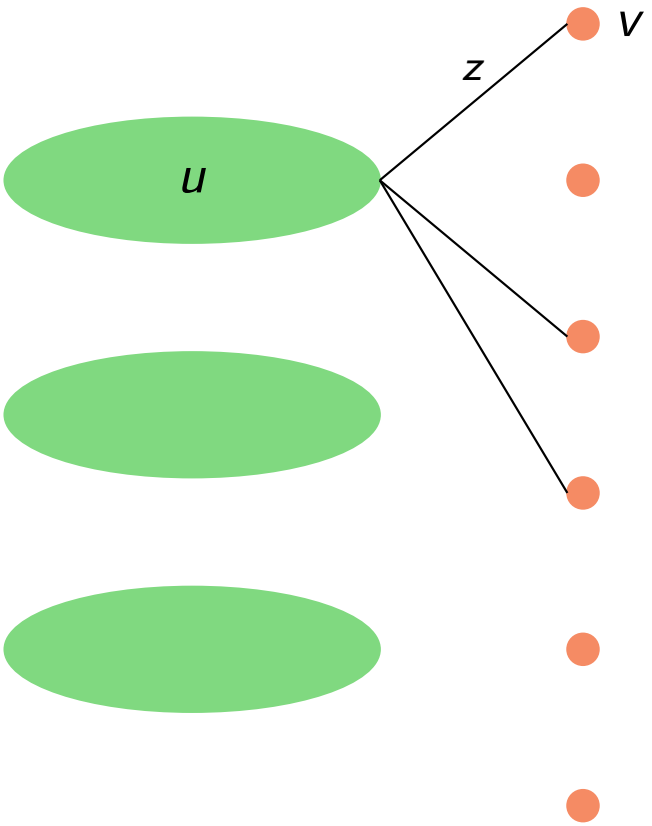

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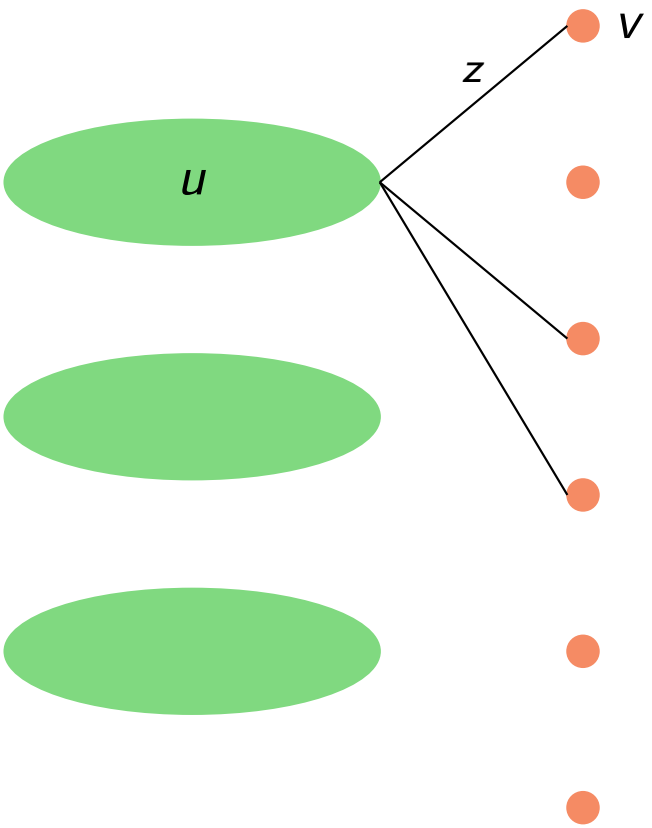
- Labels on left are functions $f_u : \{0,1\}^{d+1} \to \{0,1\}$. Labels on right are bits.
- Constraint on edge (u, v) labeled by an input z to f_u. Constraints of the form

$$f_u(z) \;=\; L(v) + c_{uv,z}.$$

Linear vs degree-d labeling



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  - $\exists$ linear functions $\{f_u\}$ satisfying ≈ 1 fraction of constraints.
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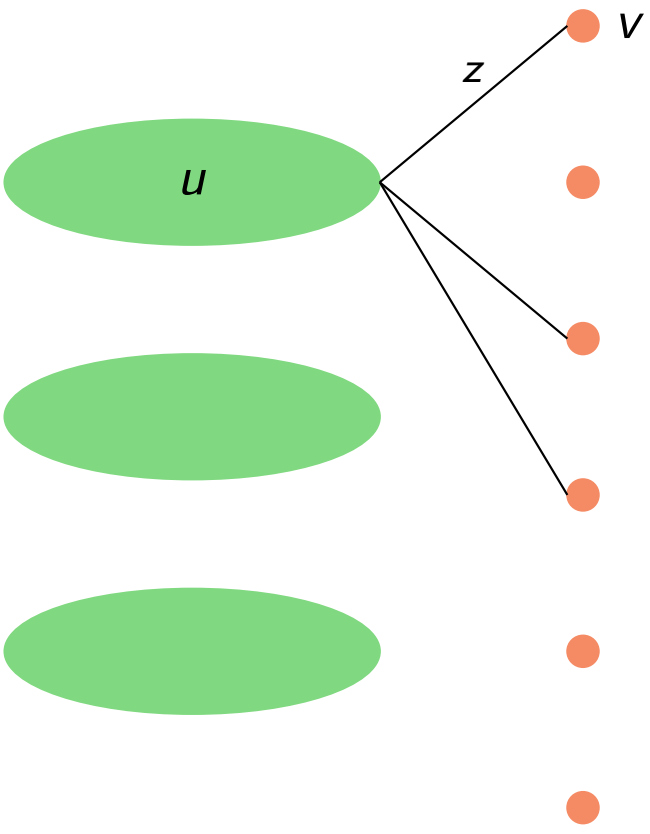

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- Strengthen via parallel repetition to combine with inner PCP.

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- Even a version of the outer PCP here with perfect completeness would be interesting. Cannot have linear functions in the first case.

Thank You

Questions?