Dense Subsets of Pseudorandom Sets

O. Reingold[1] L. Trevisan[2] M. Tulsiani[2] S. Vadhan[3]

[1]Weizmann Institute

[2]UC Berkeley

[3]Harvard University

Progressions in Subsets of Integers

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Green-Tao showed that a property of dense subsets of the integers (having progressions) also holds for the primes.

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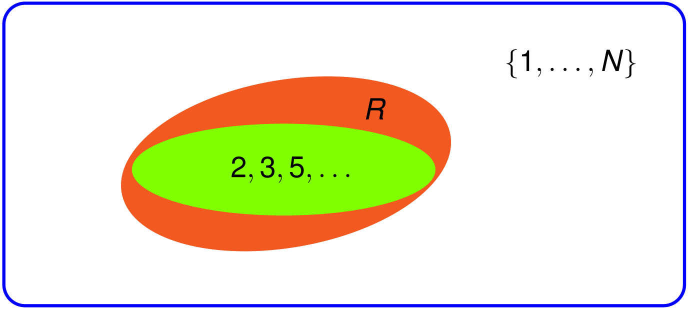

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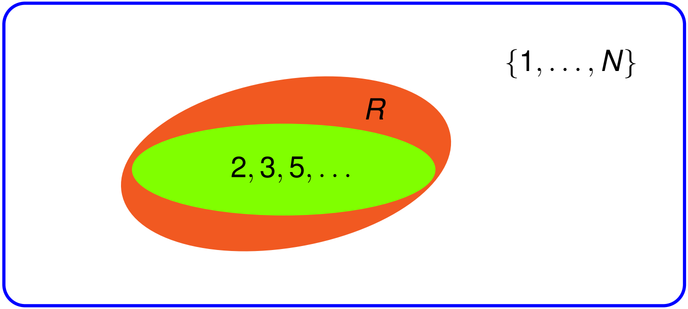

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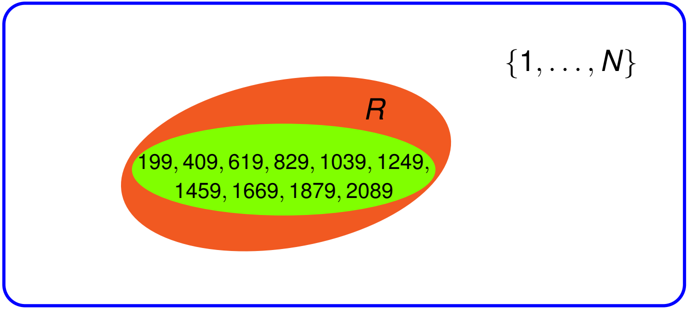
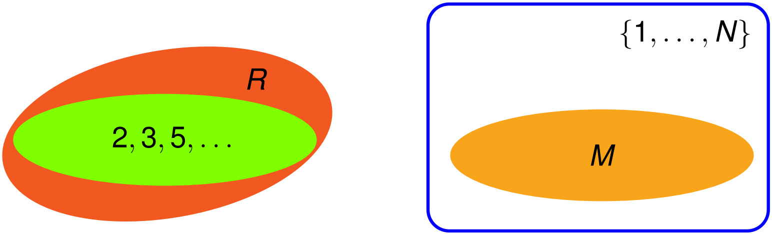


Proof of Theorem 2

- If D is a dense in a pseudorandom set R ($|D| \geq \delta|R|$), then there is a dense model set M ($|M| \geq \delta N$) indistinguishable from D.



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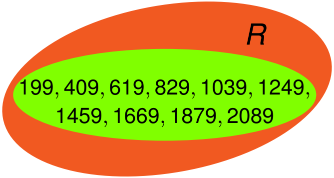

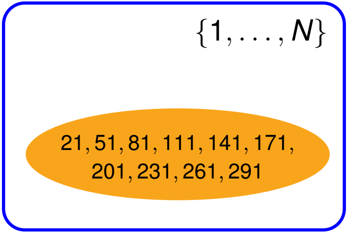

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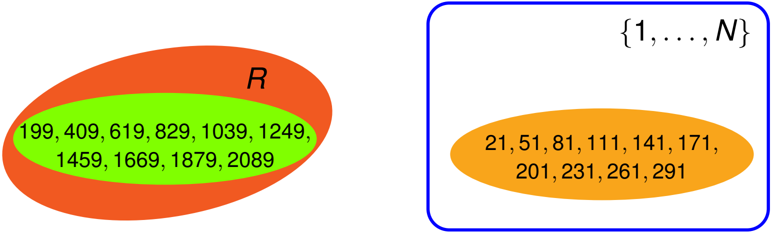

"A dense subset of a pseudorandom set has a dense model."
Can we prove this in general?

Abstracting out...

- A finite universe X (e.g. $\{1, \ldots, N\}, \{0,1\}^n$).
- A family of distinguishers $\mathcal{F} = \{f : X \to \{0,1\}\}$ (e.g. Circuits of size s).

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- Distributions A and B are ϵ-indistinguishable by $\mathcal{F}$ if

$$\forall f \in \mathcal{F}\, |\mathbb{E}f(A) - \mathbb{E}f(B)| \leq \epsilon$$

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- A is δ-dense in B if

  $$\mathbb{P}(A = x) \leq \frac{1}{\delta} \mathbb{P}(B = x)$$

  (e.g. $B = U_X$, A uniform on $\delta|X|$ elements $\implies \mathbb{P}(A = x) = \frac{1}{\delta|X|}$).

What should a "Dense Model Theorem" be?

D is δ-dense in R, R is ϵ-pseudorandom w.r.t $\mathcal{F}$.

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There is M δ-dense in U_X, ϵ-indistinguishable from D by $\mathcal{F}$.

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What should a "Dense Model Theorem" be?

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R is ϵ'-distinguishable from U_X by $\mathcal{F}'$.

Relation between (ϵ, ϵ') and $(\mathcal{F}, \mathcal{F}')$ depends on the reduction.

The Results

Theorem (Tao-Ziegler 2006)

Suppose for all M δ-dense in U_X, some function in $\mathcal{F}$ ϵ-distinguishes M and D. Then, there is a function $h : X \to \{0,1\}^n$ of the form

$$h(x) = g(f_1(x), \ldots, f_k(x)) \quad f_i \in \mathcal{F}, k = poly(1/\epsilon, 1/\delta)$$

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- Switching the quantifiers

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Proof: $\mathbb{E}Z$ is the average of $\mathbb{P}(Z \geq t)$ over $t \in (0,1)$.

In fact, $\exists t \quad \mathbb{P}(\bar{f}(D) \geq t + \epsilon/2) - \mathbb{P}(\bar{f}(M) \geq t) \geq \epsilon/2$

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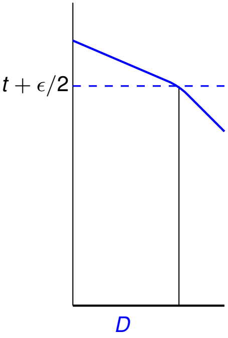

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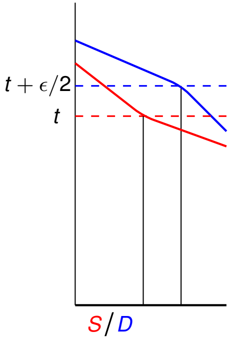

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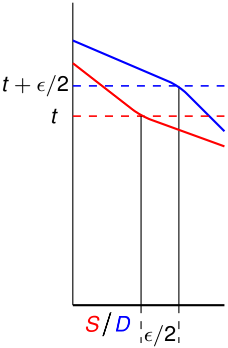

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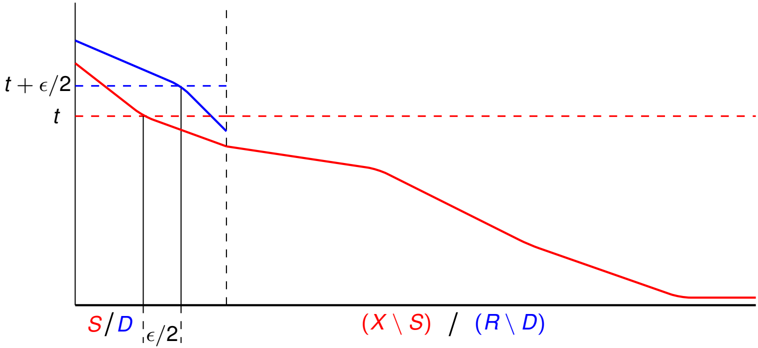

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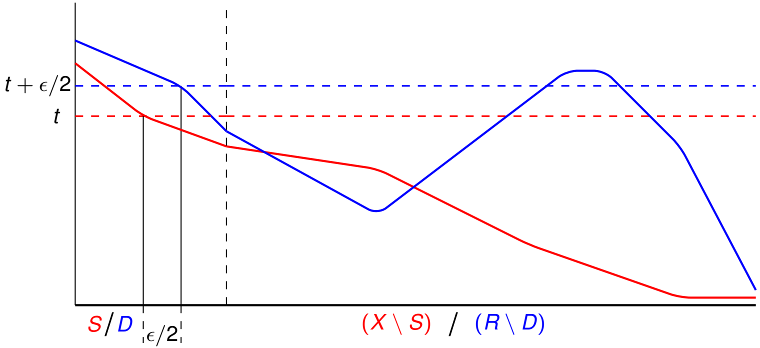

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  Sample $k = poly(1/\epsilon, \log 1/\delta)$ functions $f_1, \ldots, f_k$

$$\mathbb{P}\left(\frac{\sum f_i(R)}{k} \geq t + \epsilon/4\right) - \mathbb{P}\left(\frac{\sum f_i(U_X)}{k} \geq t + \epsilon/4\right) \geq \ \epsilon\delta/4$$

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- Note that we combine $f_1, \ldots, f_k$ only as a linear threshold function. Complexity = $O(k)$.

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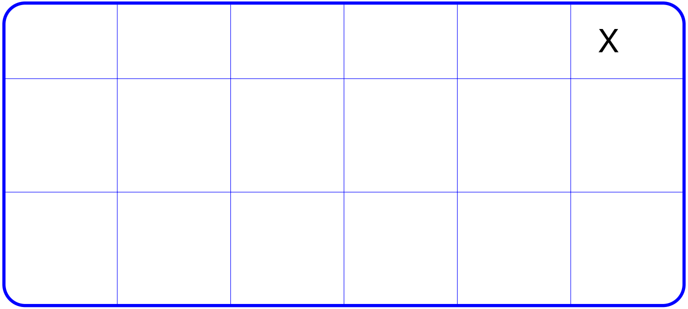

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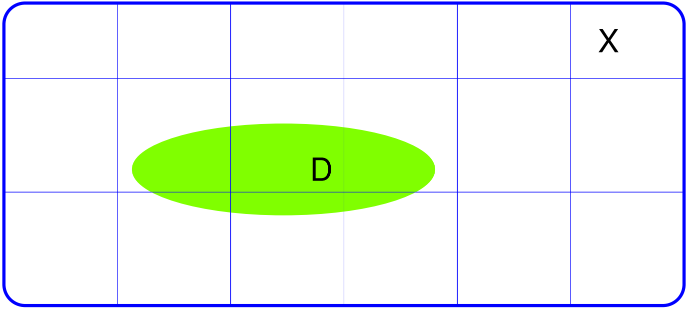

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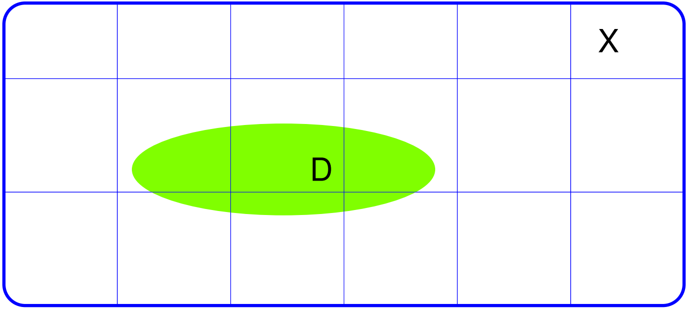

- To get M, pick whole pieces according to density of D in the piece.
- If D is distinguishable from M, then can refine partition.
- Use pseudorandomness of R to bound number of steps.

Smuggling techniques in the other direction

- We adapt the Green-Tao proof technique to prove Impagliazzo's hardcore lemma:

> If function $f : X \to \{0, 1\}$ is hard to compute correctly on more than $1 - \delta$ fraction of inputs from X then there is a set $H \subseteq X$, $|H| \geq \delta|X|$ such that f is "very hard" to compute on H.

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- Iterative partitioning gives a circuit for computing H.

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- Other applications of “ergodic arguments” in complexity theory?