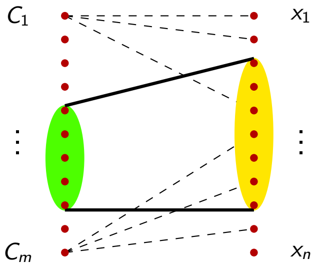
The Complexity of Somewhat Approximation Resistant Predicates



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TTI Chicago

Joint work with
Subhash Khot and Pratik Worah

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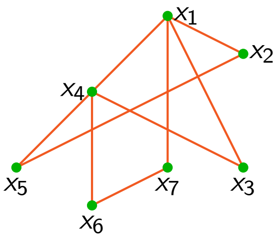

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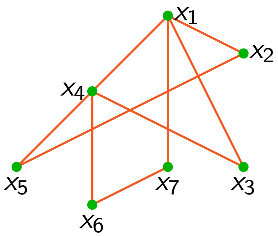

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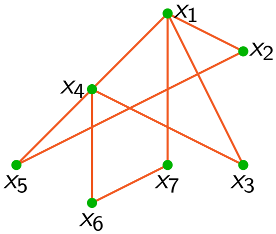

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One of the most fundamental classes of optimization problems.

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$$C_i \equiv f\left(x_{i_1} + b_1^{(i)}, \ldots, x_{i_k} + b_k^{(i)}\right)$$

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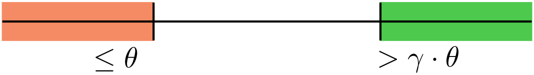

$(\gamma \geq 1)$

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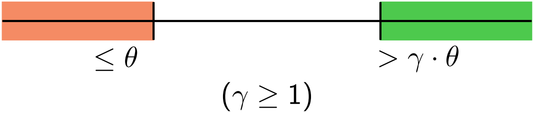

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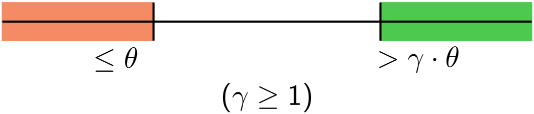

- Can solve for all $\theta \implies$ Can approximate within factor γ.
- Hard to solve for some $\theta \implies$ Hard to approximate within factor γ.

Approximation Resistance

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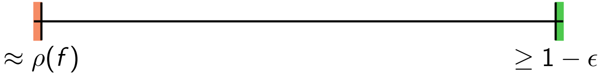

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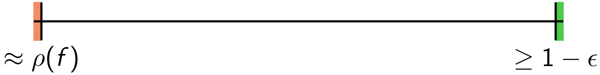

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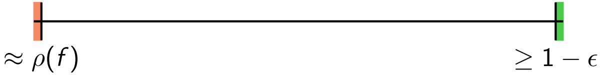

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- [ST 06*, AM 09*, Chan 12]: Approximation resistance for large classes of predicates.

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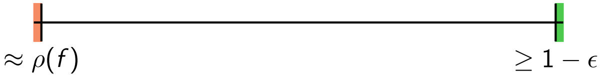

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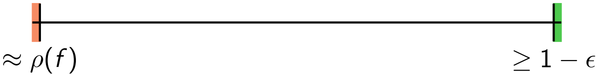
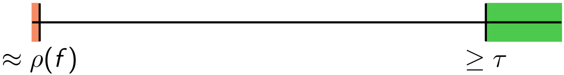


- [Håstad 07]: P is somewhat approximation resistant if for some $\tau > \rho(f)$ it is (NP-) hard to distinguish



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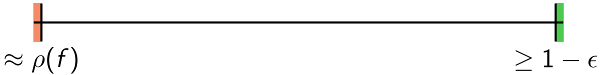

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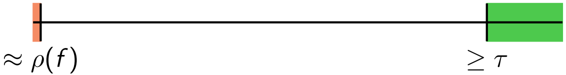

- Question: What is the maximal $\tau = \tau(f)$ for a given f?

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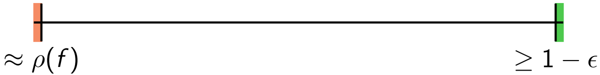

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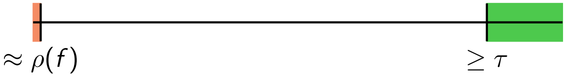

- Question: What is the maximal $\tau = \tau(f)$ for a given f?

- [Håstad 07]: Characterized $\tau(f) - \rho(f)$ up to a factor of $2^{O(k)}$.

Fourier Degree of Predicates

$$f(x) = \sum_{\alpha \in \{0,1\}^k} \widehat{f}(\alpha) \cdot (-1)^{\alpha \cdot x} = \sum_{\alpha \in \{0,1\}^k} \widehat{f}(\alpha) \cdot \chi_\alpha(x)$$

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- $\Delta(f, \mathcal{Q}) \;=\;$ min (fractional) Hamming distance from $g \in \mathcal{Q}$.

Håstad's Result

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- (Implicit) $\tau(f) - \rho(f) \geq \max_{|\alpha| \geq 3} \left|\widehat{f}(\alpha)\right|$
- There exist predicates for which $\tau(f) - \rho(f) = \Omega(1)$ but $\max_{|\alpha| \geq 3} \left|\widehat{f}(\alpha)\right| = 2^{-O(k)}$.

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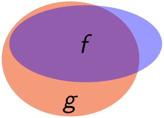

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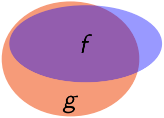

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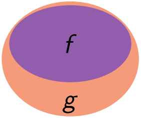

Else $f \leq g$ and

$C \cdot \delta k^3 \; \geq \; \tau(f) - \rho(f) \; \geq \; \frac{c \cdot \delta}{k^2}$

A better parameter to study for lower bounds

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- Relate $\gamma_3(f)$ to $\Delta(f, \mathcal{Q})$.

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- [FKN 02]: Relate Fourier mass above level 1 to distance from set $\mathcal{L}$ of Boolean functions with Fourier degree at most 1.

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- Both bounds generalize for γ_r and distance from degree-$(r-1)$ functions.

A class of Approximation Resistant Predicates [Chan 12]

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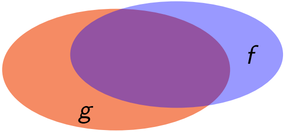
- Call these Good Subspace Predicates.

Resistance from Correlation

- Define $\mathrm{Cor}(f,g) = \dfrac{|f^{-1}(1) \cap g^{-1}(1)|}{|g^{-1}(1)|} = \dfrac{\mathbb{E}[f(x)g(x)]}{\mathbb{E}[g(x)]}$



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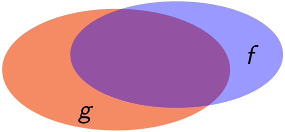

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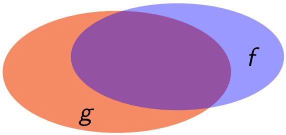

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- Will show if $\gamma_3(f)$ is large, then f correlates with a good subspace predicate.

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$$g(x) = \prod_{i=1}^{d} \left(\frac{1 + (-1)^{A_i \cdot x}}{2} \right)$$

The Fourier Spectrum of Good Subspace Predicates

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$$= \sum_{S \subseteq [d]} \frac{(-1)^{|S|}}{2^d} \cdot (-1)^{\left(\sum_{i \in S} A_i\right) \cdot x}$$

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- Will show if f does not correlate with any such g, then $\gamma_3(f)$ must be small.

A Lower Bound on Correlation

- Consider random good subspace predicate g, with $\left|g^{-1}(1)\right| = O(k^2)$ ($\dim(A) = k - 2\log k - O(1)$).

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- Non-empty Fourier coefficients of g have degree at least 3. If f has correlation at most τ with all g in random family, then

$$\gamma_3(f) \;\leq\; O(k^2)\cdot\left[\tau^2 - \rho(f)^2\right]$$

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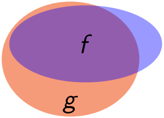
- Large $\gamma_3(f)$ implies high correlation with some good g.

Our Results

Let $\Delta(f, \mathcal{Q}) = \delta$.

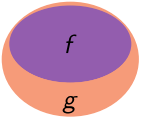
- If $\delta \geq \frac{1}{k^3}$, then $1 \;\geq\; \tau(f) - \rho(f) \;\geq\; \frac{c}{k^5}$

- If $\delta \leq \frac{1}{k^3}$, let $g \in \mathcal{Q}$ be unique closest function.



If $f^{-1}(1) \cap g^{-1}(0) \neq \emptyset$, then

$$1 \;\geq\; \tau(f) - \rho(f) \;\geq\; \frac{c}{k}$$



Else $f \leq g$ and

$$C \cdot \delta k^3 \;\geq\; \tau(f) - \rho(f) \;\geq\; \frac{c \cdot \delta}{k^2}$$

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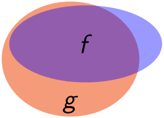

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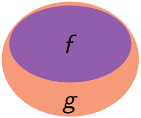

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Thank You

Questions?