A Characterization of Strong Approximation Resistance

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Joint work with
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Algorithm

zero-sum game

Hardness

value = 0
Max-k-CSP

- Boolean variables, \( m \) constraints (each on \( k \) variables)
- Satisfy as many as possible.

Max-3-SAT

\[ x_1 \lor x_2 \lor x_3 \lor \ldots \]

Max-Cut

\[ x_1 x_2 x_3 x_4 \ldots \]
Max-k-CSP

- $n$ Boolean variables, $m$ constraints (each on $k$ variables)
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Max-3-SAT

\[
x_1 \lor x_{22} \lor \overline{x}_{19} \\
x_3 \lor \overline{x}_9 \lor x_{23} \\
x_5 \lor \overline{x}_7 \lor \overline{x}_9 \\
\vdots
\]

One of the most fundamental classes of optimization problems.
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\[\vdots\]

Max-Cut

[Diagram of a graph with vertices labeled $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ connected by edges.]
Max-k-CSP

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\begin{align*}
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& \vdots
\end{align*}
\]

Max-Cut

\[
\begin{align*}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{align*}
\]

$\begin{align*}
x_1 & \not= x_2 \\
x_2 & \not= x_5 \\
x_3 & \not= x_4 \\
& \vdots
\end{align*}$
Max-k-CSP

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    \vdots
\end{align*}
\]

Max-Cut

\[
\begin{align*}
    x_1 & \neq x_2 \\
    x_2 & \neq x_5 \\
    x_3 & \neq x_4 \\
    \vdots
\end{align*}
\]

One of the most fundamental classes of optimization problems.
Max-k-CSP

Max-3-XOR: Linear equations modulo 2 (in ±1 variables)
Max-k-CSP

**Max-3-XOR:** Linear equations modulo 2 (in ±1 variables)

\[
\begin{align*}
   x_5 \cdot x_9 \cdot x_{16} &= 1 \\
   x_6 \cdot x_{12} \cdot x_{22} &= -1 \\
   x_7 \cdot x_8 \cdot x_{15} &= -1 \\
   \vdots
\end{align*}
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\[
\begin{align*}
    x_5 \cdot x_9 \cdot x_{16} &= 1 \\
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Max-k-CSP($f$): Given predicate $f: \{-1, 1\}^k \rightarrow \{0, 1\}$. Each constraint is $f$ applied to some $k$ (possibly negated) variables.
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Max-k-CSP\((f)\): Given predicate \(f : \{-1, 1\}^k \rightarrow \{0, 1\}\). Each constraint is \(f\) applied to some \(k\) (possibly negated) variables.

\[
C_i \equiv f (x_{i_1} \cdot b_1^{(i)}, \ldots, x_{i_k} \cdot b_k^{(i)} )
\]
Approximating Max-k-CSP

Relax the problem of finding maximum fraction of constraints satisfiable.
Approximating Max-k-CSP

Relax the problem of finding \textbf{maximum fraction} of constraints satisfiable.

\[
\begin{align*}
\leq \theta & \quad \leq \theta \\
\quad & \quad > \gamma \cdot \theta \\
(\gamma \geq 1) & \quad > \gamma \cdot \theta
\end{align*}
\]
Approximating Max-k-CSP

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- Can solve for all $\theta$ $\implies$ Can approximate within factor $\gamma$. (\(\gamma \geq 1\))
Approximating Max-k-CSP

Relax the problem of finding maximum fraction of constraints satisfiable.

\[ \leq \theta \quad \text{and} \quad > \gamma \cdot \theta \quad (\gamma \geq 1) \]

- Can solve for all \( \theta \) \( \implies \) Can approximate within factor \( \gamma \).
- Hard to solve for some \( \theta \) \( \implies \) Hard to approximate within factor \( \gamma \).
- Let $\rho(f) = \mathbb{E}_x[f(x)]$ be the fraction of constraints satisfied by a random assignment.
Approximation Resistance

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- $\rho(3\text{-SAT}) = 7/8$, $\rho(3\text{-XOR}) = 1/2$
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- $f$ is approximation resistant if it is (NP/UG-) hard to distinguish

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\leq \rho(f) + \epsilon \quad \quad \geq 1 - \epsilon
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- $f$ is **approximation resistant** if it is (NP/UG-) hard to distinguish


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- Captures the notion of when is it hard to do better than a random assignment.
(Sufficient) Conditions for Approximation Resistance

- [Håstad 01]: k-SAT and k-XOR are approximation resistant.
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- [AK 13∗]: Characterization when $f$ is even and instance is required to be $k$-partite.
- $f$ is approximation resistant if it is (NP/UG-) hard to distinguish

\[ \leq \rho(f) + \epsilon \quad \geq 1 - \epsilon \]

- When is hard to do anything different from a random assignment.
- Equivalent to approximation resistance for odd predicates. Almost all previous results prove strong approximation resistance.
- f is approximation resistant if it is (NP/UG-) hard to distinguish

\[ \leq \rho(f) + \epsilon \quad \text{and} \quad \geq 1 - \epsilon \]

- f is strongly approximation resistant if it is (NP/UG-) hard to distinguish

\[ [\rho(f) - \epsilon, \rho(f) + \epsilon] \quad \text{and} \quad \geq 1 - \epsilon \]
Strong Approximation Resistance

- $f$ is approximation resistant if it is (NP/UG-) hard to distinguish
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Strong Approximation Resistance

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A partial characterization by [Rag 08] and [RS 09]

- [Rag 08*]: f is approximation resistant iff $\forall \epsilon > 0$ there exists a $1 - \epsilon$ vs. $\rho(f) + \epsilon$ integrality gap instance for a certain SDP.
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- But what properties of $f$ give rise to gap instances?

- Is it just properties of $f$ or is the topology of the instance also important? (Hint: Just $f$)
The Austrin-Mossel condition in a new language

- For a distribution $\mu$ on $\{-1, 1\}^k$, let $\zeta(\mu) \in \mathbb{R}^{k+\binom{k}{2}}$ denote the vector of first and second moments

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\zeta_i = \mathbb{E}_{x \sim \mu}[x_i] \quad \zeta_{ij} = \mathbb{E}_{x \sim \mu}[x_i \cdot x_j]
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- Let $C(f)$ be the convex polytope

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- [AM 09*]: $f$ is (strongly) approximation resistant if $0 \in C(f)$.

- Our condition is in terms of existence of a measure $\Lambda$ on $C(f)$ with certain symmetry properties.
Transformations of a measure $\Lambda$ on $C(f)$

- Each $\zeta \in C(f)$ can be transformed by:
Transformations of a measure $\Lambda$ on $C(f)$

- Each $\zeta \in C(f)$ can be transformed by:
  - Permuting the underlying $k$ variables by a permutation $\pi$

\[
(\zeta_\pi)_i = \zeta_{\pi(i)} \quad (\zeta_\pi)_{ij} = \zeta_{\pi(i)\pi(j)}
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- For $S \subseteq [k]$, $\pi : S \rightarrow S$, $b \in \{-1, 1\}^S$, let $\Lambda_{S, \pi, b}$ denote the measure obtained by transforming each point in support of $\Lambda$ as above.
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- For $S \subseteq [k]$, $\pi : S \rightarrow S$, $b \in \{-1, 1\}^S$, let $\Lambda_{S,\pi,b}$ denote the measure obtained by transforming each point in support of $\Lambda$ as above.

- If $\Lambda$ is supported only on 0, then so is each $\Lambda_{S,\pi,b}$. If $\Lambda$ is supported only on (say) $(1,\ldots,1)$ then $\Lambda_{[k],\text{id},b}$ is supported only on the point $(b_1, \ldots, b_k, b_1 \cdot b_2, \ldots, b_{k-1} \cdot b_k)$
Our Characterization

- Recall that \( f : \{-1, 1\}^k \rightarrow \{0, 1\} \) can be written as

\[
f(x) = \sum_{S \subseteq [k]} \hat{f}(S) \cdot \prod_{i \in S} x_i = \rho(f) + \sum_{t=1}^{k} \sum_{|S|=t} \hat{f}(S) \cdot \prod_{i \in S} x_i
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$$f(x) = \sum_{S \subseteq [k]} \hat{f}(S) \cdot \prod_{i \in S} x_i = \rho(f) + \sum_{t=1}^{k} \sum_{|S|=t} \hat{f}(S) \cdot \prod_{i \in S} x_i$$

- [KTW 13*]: $f$ is strongly approximation resistant if and only if there exists a measure $\Lambda$ on $C(f)$ such that for all $t = 1, \ldots, k$

$$\sum_{|S|=t} \sum_{\pi : S \rightarrow S} \sum_{b \in \{-1, 1\}^S} \hat{f}(S) \cdot \left( \prod_{i \in S} b_i \right) \cdot \Lambda_{S, \pi, b} \equiv 0$$
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- [K_TW 13*]: $f$ is strongly approximation resistant if and only if there exists a measure $\Lambda$ on $C(f)$ such that for all $t = 1, \ldots, k$

$$\sum_{|S|=t} \sum_{\pi:S \rightarrow S} \sum_{b \in \{-1, 1\}^S} \hat{f}(S) \cdot \left( \prod_{i \in S} b_i \right) \cdot \Lambda_{S, \pi, b} \equiv 0$$

- If $|S| = t$, then $\Lambda_{S, \pi, b}$ is a measure on $\mathbb{R}^{t+\binom{t}{2}}$. For each $t$, above expression is a linear combination of such measures.
Proof Structure

- No good \( \Lambda \) exists
- Good \( \Lambda \) exists

Standard PCP ideas

Hardness of zero-sum game

Algorithm hardness value > 0

Algorithm hardness value = 0
Proof Structure

No good \( \Lambda \) exists

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Standard PCP ideas

Hardness
Proof Structure

- No good $\Lambda$ exists
- Good $\Lambda$ exists

For Good $\Lambda$ exists:
- Algorithm
- Hardness: zero-sum game
- Value: $> 0$

For No good $\Lambda$ exists:
- Algorithm
- Hardness: zero-sum game
- Value: $= 0$

Standard PCP ideas lead to Hardness.
Proof Structure

No good \( \Lambda \) exists

\[ \text{Algorithm} \]
\[ \text{zero-sum game} \]
\[ \text{value} > 0 \]

Good \( \Lambda \) exists

\[ \text{Algorithm} \]
\[ \text{zero-sum game} \]
\[ \text{value} = 0 \]

Standard PCP ideas

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No good \( \Lambda \) exists

Good \( \Lambda \) exists

Standard PCP ideas

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zero-sum game

value > 0

Algorithm

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Proof Structure

- **No good $\Lambda$ exists**
  - Algorithm
  - Zero-sum game
  - Value $> 0$

- **Good $\Lambda$ exists**
  - Algorithm
  - Zero-sum game
  - Value $= 0$

- Standard PCP ideas

- Hardness
The (infinite) two-player game

- Similar game also used by O’Donnell and Wu for Max-Cut.
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- Hardness player tries to design an integrality-gap instance. Each constraint has local distribution $\mu$ with moments given by $\zeta(\mu)$. Plays measure $\Lambda$ on $C(f)$ (corresponds to instance).
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- Algorithm player tries to round by first projecting to random $d$-dimensional Gaussian. Plays rounding strategy $\psi : \mathbb{R}^d \rightarrow \{-1, 1\}$. ($d = k + 1$ suffices)
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- Value $= |\rho(f) - \text{Expected fraction of constraints satisfied by } \psi|$
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- Value $> 0$ implies (a distribution over) rounding strategies which show that predicate is not strongly approximation resistant. (since every instance corresponds to a $\Lambda$)
Value of the game

- A random constraint in the instance corresponds to $\zeta \sim \Lambda$. 
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- Projecting gives Gaussians $y_1, \ldots, y_k$ with correlation matrix corresponding to $\zeta$ ($y_1, \ldots, y_k \sim N(\zeta)$).

- Expected fraction of constraints satisfied

$$\mathbb{E}_{\zeta \sim \Lambda} \mathbb{E}_{y_1, \ldots, y_k \sim N(\zeta)} \left[ f(\psi(y_1), \ldots, \psi(y_k)) \right]$$

$$= \rho(f) + \mathbb{E}_{\zeta \sim \Lambda} \mathbb{E}_{y_1, \ldots, y_k \sim N(\zeta)} \left[ \sum_{S \neq \emptyset} \hat{f}(S) \cdot \prod_{i \in S} \psi(y_i) \right]$$
Value of the game

- A random constraint in the instance corresponds to $\zeta \sim \Lambda$.

- When Algorithm player tries to round SDP solution, for she sees vectors with inner products according to $\zeta$.

- Projecting gives Gaussians $y_1, \ldots, y_k$ with correlation matrix corresponding to $\zeta$ ($y_1, \ldots, y_k \sim N(\zeta)$).

- Expected fraction of constraints satisfied

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$$
Obtaining conditions on $\Lambda$ when value = 0

- Value $= \left| E_{\zeta \sim \Lambda} E_{y_1, \ldots, y_k \sim N(\zeta)} \left[ \sum_{S \neq \emptyset} \hat{f}(S) \cdot \prod_{i \in S} \psi(y_i) \right] \right|$. 
Obtaining conditions on $\Lambda$ when value $= 0$

- Value $= |\mathbb{E}_{\zeta \sim \Lambda} \mathbb{E}_{y_1, \ldots, y_k \sim \mathcal{N}(\zeta)} \left[ \sum_{S \neq \emptyset} \hat{f}(S) \cdot \prod_{i \in S} \psi(y_i) \right]|$.

- There exists (distribution over) $\Lambda$ which gives value 0 for all $\psi$.
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- All coefficients must be 0. Coefficients are linear combinations of integrals of $\Lambda S, \pi, b$ w.r.t. some Gaussian densities.

- Need to conclude integrals are zero only if the corresponding linear combinations are 0. Degree $t$ coefficients give condition at level $t$.

- Bulk of the work in analyzing sequence of finite games and coefficients of corresponding polynomials.
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Concluding Remarks

- We also characterize
  - Approximation resistance for odd predicates (including threshold functions passing through origin).
  - Approximation resistance for $k$-partite instances (all predicates).
  - Sherali-Adams LP gaps for $\omega(1)$ levels (all predicates).
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- **Problem**: Strong Approximation Resistance vs. Approximation Resistance.
Thank You

Questions?