1 Mapping between vector spaces

1.1 Basic definitions

Definition 1.1 (Linear Map)  We call a mapping $\Phi$ from vector space $V$ to $W$ a linear map if

1. $\forall v_1, v_2 \in V, \quad \Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2)$
2. $\forall \alpha \in \mathbb{F}, \forall v \in V, \quad \Phi(\alpha \cdot v) = \alpha \cdot \Phi(v)$

$\Phi$ is an isomorphism if it is a bijection.

Exercise 1.2  Show that $\Phi(0) = 0$ for any linear map $\Phi$.

Exercise 1.3  Prove that an isomorphism maps an linearly independent set to an linearly independent set.

Exercise 1.4  Show that for an isomorphism $\Phi$, $\text{rank}(v_1, \ldots, v_n) = \text{rank}(\Phi(v_1), \ldots, \Phi(v_n))$.

Definition 1.5 (Kernel and Image)  Given $\Phi : V \to W$, the kernel and image of $\Phi$ are defined as

$$
\ker(\Phi) := \{ v \in V : \Phi(v) = 0 \} \quad \text{and} \quad \text{Im}(\Phi) := \{ \Phi(v) : v \in V \}.
$$

Exercise 1.6  $\ker(\Phi)$ and $\text{Im}(\Phi)$ are vector spaces.

The dimension of ($\text{Im}(\Phi)$) is called then rank of $\Phi$ and the dimension of $\ker(\Phi)$ nullity of $\Phi$.

Theorem 1.7 (Rank-Nullity Theorem)  $\dim(\ker(\Phi)) + \dim(\text{Im}(\Phi)) = \dim(V)$.

Exercise 1.8  Let $V = \mathbb{R}^\leq n[x]$ and $\Phi = d/dx$. What are $\ker(\Phi)$ and $\text{Im}(\Phi)$?

Proof of theorem:  Let $\dim(V) = n$, and $\dim(\ker(\Phi)) = k$. Let $v_1, \ldots, v_k$ be a basis for $\ker(\Phi)$. Complete $v_1, \ldots, v_k$ to $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ which is a basis for $V$. We claim that $\Phi(v_{k+1}), \ldots, \Phi(v_n)$ form a basis for $\text{Im}(\Phi)$. We first prove that they are linearly independent.

Claim 1.9  $\Phi(v_{k+1}), \ldots, \Phi(v_n)$ are linearly independent in $W$. 
**Proof of claim:** Let \( \alpha_{k+1} \cdot \Phi(v_{k+1}) + \cdots + \alpha_n \cdot \Phi(v_n) = 0 \). Then \( \Phi(\alpha_{k+1}v_{k+1} + \cdots + \alpha_nv_n) = 0 \). This implies

\[
\alpha_{k+1}v_{k+1} + \cdots + \alpha_nv_n \in \ker(\Phi) = \text{span}(v_1, \ldots, v_k).
\]

However, since \( v_1, \ldots, v_n \) are linearly independent, this gives that \( \alpha_{k+1}, \ldots, \alpha_n = 0. \) \( \square \)

Now left to show that \( \text{Span}(\Phi(v_{k+1}), \ldots, \Phi(v_n)) = \text{Im}(\Phi) \). Let \( w \in \text{Im}(\Phi) \). Then \( w = \Phi(v_1) + \cdots + \Phi(v_n) \) for \( v_1, \ldots, v_n \in \mathbb{F} \).

\[
\Phi(\alpha_1v_1 + \cdots + \alpha_nv_n) = \underbrace{\alpha_1\Phi(v_1) + \cdots + \alpha_k\Phi(v_k) + \alpha_{k+1}\Phi(v_{k+1}) + \cdots + \alpha_n\Phi(v_n)}_{=0}
\]

Therefore \( \text{Im}(\Phi) \subseteq \text{Span}(\Phi(v_{k+1}), \ldots, \Phi(v_n)) \). Also, for any \( w \in \text{Span}(\Phi(v_{k+1}), \ldots, \Phi(v_n)) \), of the form \( w = \alpha_{k+1}\Phi(v_{k+1}) + \cdots + \alpha_n\Phi(v_n) \), we have that

\[
w = \alpha_{k+1}\Phi(v_{k+1}) + \cdots + \alpha_n\Phi(v_n) = \Phi(\alpha_{k+1}v_{k+1} + \cdots + \alpha_nv_n),
\]

which shows that \( \text{Span}(\Phi(v_{k+1}), \ldots, \Phi(v_n)) \subseteq \text{Im}(\Phi) \). \( \blacksquare \)

**Exercise 1.10** Let \( b_1, \ldots, b_n \) be a basis for \( V \). Then there is a unique linear map \( \Phi \) such that \( \Phi(b_1) = w_1, \ldots, \Phi(b_n) = w_n \)

**Exercise 1.11**

1. An isomorphism maps a basis to a basis.

2. Any two vector spaces (over same set of scalars) with equal (finite) dimension are isomorphic (i.e. there exists an isomorphism between them)

### 1.2 Linear map as a matrix

**Proposition 1.12** A \( m \times n \) matrix \( A \) over \( \mathbb{F} \) is a linear map from \( \mathbb{F}^m \) to the column space of \( A \)

**Proof:** \( A(x_1 + x_2) = Ax_1 + Ax_2 \) and \( A(\alpha x) = \alpha Ax. \) \( \blacksquare \)

Therefore, applying rank-nullity theorem to the matrix, since \( \text{Im}(A) \) is the column-space of \( A \) and \( \ker(A) := \{ x \mid Ax = 0 \}, \text{rank}(A) + \text{dim}(\ker(A)) = n. \)

Let \( \Phi : V \rightarrow W \) be a linear map. Let \( \{e_1, \ldots, e_n\} \) be a basis for \( V \), and \( \{f_1, \ldots, f_m\} \) for \( W \). Then given \( v \in V, \exists \) unique \( \alpha_1, \ldots, \alpha_n \) such that \( v = \sum \alpha_ie_i \).

Let \([v]_\pi \) denote \( \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \). Since for \( \Phi(e_i) \in W, \exists \beta_{1i}, \ldots, \beta_{mi} \) such that \( \Phi(e_i) = \beta_{1i}f_1 + \cdots + \beta_{mi}f_m. \)

Then we write
$$A_{\Phi} = \begin{bmatrix} \Phi(e_1) & \cdots & \Phi(e_n) \end{bmatrix} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{bmatrix}$$

If we apply $A_{\Phi}$ to $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, we get

$$A_{\Phi} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1[\Phi(e_1)] + \cdots + \alpha_n[\Phi(e_n)]$$

$$= [\Phi(\alpha_1e_1 + \cdots + \alpha_ne_n)]$$

Thus if $[v]_{\pi} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, $A_{\Phi}[v]_{\pi} = [\Phi(v)]_{\pi}$

**Exercise 1.13** Write $\Phi = d/dx$ on $\mathbb{R}^{\leq n}[x]$ as a matrix.

### 1.3 Application: Linear equations

Consider following system of linear equations.

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

We can simply write it as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

We want to know when a system has a solution. This exactly corresponds to saying whether $b$ is in $\text{span}(A^{(1)}, \ldots, A^{(n)})$.

**Exercise 1.14** $b \in \text{span}(A^{(1)}, \ldots, A^{(n)})$ if and only if $\text{rank}(A) = \text{rank}(A|b)$

If $b = 0$, we call such system homogeneous system, and we ask $\exists? x \neq 0$ such that $Ax = 0$.

**Exercise 1.15** Non-zero solution exists if and only if $\text{rank}(A) < n$. 

3
Exercise 1.16 (Graph coloring continued)  Show that the following graphs are 2-colorable (bi-partite).

1. $n \times m$ grid

2. Hypercubes ($V = \{0, 1\}^n$, $E = \{(x, y) \mid x and y differ in exactly one position\}$)