1 Linear Transformation continued

Recall that $\Phi: V \rightarrow W$ with $\{e_1, \ldots, e_n\}$ as a basis for $V$ and $\{f_1, \ldots, f_m\}$ as basis for $W$. Then we can write $M_\Phi$ as following

$$M_\Phi = [\Phi]_{\epsilon, \eta} = \begin{pmatrix}
\Phi(e_1)_{\cdot, f_1} & \cdots & \Phi(e_n)_{\cdot, f_1} \\
\vdots & \ddots & \vdots \\
\Phi(e_1)_{\cdot, f_m} & \cdots & \Phi(e_n)_{\cdot, f_m}
\end{pmatrix}$$

**Example 1.1** Let $\Phi$ be a rotation by angle $\theta$, under usual basis. Then we can write the linear transformation as

$$M_\Phi = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}$$

**Definition 1.2 (Composition of Linear Map)** Let $\Phi_1: V \rightarrow W$ and $\Phi_2: W \rightarrow Z$. Then $\Phi_2 \circ \Phi_1: V \rightarrow Z$ where

$$\Phi_2 \circ \Phi_1(v) = \Phi_2(\Phi_1(v))$$

**Exercise 1.3**

1. Prove $[\Phi_2 \circ \Phi_1]_{\pi, \tau} = [\Phi_2]_{\tau, \eta} \cdot [\Phi_1]_{\eta, \pi}$
2. Prove $\text{rank}(\Phi) = \text{rank}([\Phi]_{\epsilon, \eta})$

2 Determinant

**Definition 2.1 (Permutation)** $\pi: [n] \rightarrow [n]$ is called permutation if $\pi$ is a bijective map. The set of all permutations is denoted as $S_n$.

**Definition 2.2 (Inversion)** Inversion on a given permutation $\pi$ is defined as

$$\text{Inv}(\pi) = \lvert \{(i, j) \mid i < j, \pi(i) > \pi(j)\} \rvert$$

And also

$$\text{sgn}(\pi) = (-1)^{\text{Inv}(\pi)}$$

**Exercise 2.3** Let $\sigma, \pi \in S_n$, then $\text{sgn}(\sigma \circ \pi) = \text{sgn}(\sigma) \cdot \text{sgn}(\pi)$
Claim 2.4

\[ sgn(\pi) = \prod_{i<j} \left( \frac{\pi(i) - \pi(j)}{i - j} \right) \]

**Proof:** First, assume that we will get some number of absolute value 1. Then only the sign of each term matters (that is whether it is inverted or not). Therefore it will be exactly the same quantity as \( sgn(\pi) \).

Now the absolute value should be 1, since all pairs \((i, j)\) should appear on both numerator and denominator.

Definition 2.5 (Transposition) Transposition is a swap of two elements.

Exercise 2.6 Prove that every permutation can be written as a composition of transpositions.

Exercise 2.7 If \( \pi \) is a transposition, then \( sgn(\pi) = -1 \)

Exercise 2.8 Prove that every permutation can be written as a composition of “neighbor swaps” (transposition of \((i, i+1)\))

Definition 2.9 (Determinant) \( A \in \mathbb{R}^{n \times n} \) or \( A \in M_n(\mathbb{R}) \). Then

\[ det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \]

Example 2.10

\[ det \left( \begin{array}{cc} a & c \\ b & d \end{array} \right) = ad - bc \]

Exercise 2.11 Prove that if a matrix \( A \) is a “triangular” matrix then \( det(A) \) is the product of diagonal entries.

Exercise 2.12 Compute

1. \( det(A) \) where \( a_{i,i} = a \) and \( a_{i,j} = b \) if \( i \neq j \).
2. Determinant of Vandermonde matrix : \( a_{i,j} = x_i^j \)
3. \( det(A) \) where \( a_{i,i} = 1, a_{i,i+1} = -1 \) and \( a_{i+1,i} = 1 \) and 0 otherwise.

Proposition 2.13 Let \( A_{i,j} \) be a \((n-1) \times (n-1)\) matrix obtained by deleting \( i \)th row and \( j \)th column. Then

\[ det(A) = \sum_{j=1}^{n} a_{i,j}(-1)^{i+j}det(A_{i,j}) \]

Exercise 2.14 If two columns of \( A \) are equal, then \( det(A) = 0 \).

Proposition 2.15 Elementary operations do not change the determinant.