1 More on Adjacency Matrices

Recall that we have $G = (V, E)$ and its adjacency matrix $A$ and eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ and $\sum \mu_i = \text{tr}(A) = 0$.

Exercise 1.1

1. Show that the following holds

$$\frac{1}{n} \sum_{i \in V} \deg(i) \leq \mu_1 \leq \max_{i \in V} \deg(i)$$

2. If $G$ has a positive eigenvector with eigenvalue $\lambda$, then for all other eigenvalues $\mu$, with $|\mu| \leq \lambda$.

3. If $G$ is connected, then $A_G$ has a positive eigenvector with eigenvalue $\mu_1$. (Use Rayleigh quotient)

Exercise 1.2 If $G$ has $A_G$ with eigenvalues, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, then $G$ can be colored with $\lceil \mu_1 \rceil + 1$ colors.

This exercise was discussed in class. We follow the same proof scheme as before, for showing that a graph with maximum degree $d$ can be colored with $d + 1$ colors. We proceed by induction on the number of vertices in $G$. The case with $n = 1$ is trivial since the only eigenvalue is 0 and the graph can be colored with 1 color.

For the case with $n$ vertices, we know (from the previous exercise) that $\mu_1 \geq \frac{1}{n} \sum_{i \in V} \deg(i)$. Thus, there must be a vertex $i$ with degree at most $\mu_1$. Since degrees are integers, we must have $\deg(i) \leq \lfloor \mu_1 \rfloor$. Consider the graph $G'$ (on $n - 1$ vertices) obtained by removing the vertex $i$ from $G$. Use Rayleigh quotients to prove that if $\nu_1$ is the largest eigenvalue of $G'$, then $\nu_1 \leq \mu_1$. By induction, $G'$ can be colored with $\lfloor \nu_1 \rfloor + 1 \leq \lfloor \mu_1 \rfloor + 1$ colors. Since the vertex $i$ we removed has at most $\lfloor \mu_1 \rfloor$ neighbors, we can assign it a color which is different from the colors of all its neighbors. This gives a valid coloring of $G$.

Exercise 1.3 Suppose $G'$ is generated from $G$ via removing a vertex. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of $G$ and let $\nu_1 \geq \cdots \geq \nu_{n-1}$ be the eigenvalues of $G'$. Then use Rayleigh quotients to show that

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \ldots \geq \nu_{n-1} \geq \mu_n.$$
2 Random Walk on Graphs

Given a starting vertex $i_0 \in V$, a simple random walk on the graph $G = (V, E)$ is the following process:

- Start at the given vertex $i_0$.
- At each step, pick a random neighbor of the current vertex and move to neighbor vertex.

Consider a vector $p(t)$ where $p(t)(j)$ is supposed to denote the chance that the random walk after $t$ steps, is at vertex $j$. If the starting vertex is $i$, then $p(0) \in \mathbb{R}^n$ is a vector with $p(0)(j) = 1$ if $j = i_0$ and 0 otherwise. In general a probability distribution $p$ over vertices must have the property that $\sum_j p(j) = 1$ and $p(j) \geq 0 \forall j \in V$.

To understand the distribution $p(t+1)$ in terms of $p(t)$, we note that if at step $t + 1$ we are to land at a vertex $i$, then at step $t$ we must be at some $j$ which is a neighbor of $j$ (which we denote by $j \sim i$). We can then write

$$p(t+1)(i) = \sum_{j \sim i} p(t)(j) \cdot \frac{1}{\deg(j)}.$$ 

In matrix form, this gives

$$p(t+1) = M \cdot p(t),$$

where $M$ is a matrix with entries

$$M_{ij} = \begin{cases} 
\frac{1}{\deg(j)} & \text{if } i \sim j \\
0 & \text{otherwise}
\end{cases}.$$ 

Note that $M = A_G D^{-1}$ where $D$ is a diagonal matrix with entries $D_{ii} = \deg(i)$ and $A$ is the adjacency matrix. The matrix $M$ is also referred to as the diffusion matrix for the simple random walk on $G$. We can use linear algebra to analyze random walks once we notice that the distribution after $t$ steps can be written as

$$p(t) = M^t \cdot p(0).$$

In particular, we will be interested in the question of how fast the random walk reaches a stationary distribution i.e., a distribution which does not change as the random walk proceeds.

**Definition 2.1 (Stationary distribution)** A distribution $\pi$ is a stationary distribution for a random walk with diffusion matrix $M$ if

$$M \cdot \pi = \pi.$$ 

Thus, $\pi$ is simply a non-negative eigenvector of $M$ with eigenvalue 1, which is multiplied by an appropriate positive constant to ensure that $\sum_i \pi(i) = 1$. For all graphs, the random walk on $G$ will have a stationary distribution (we will prove this in the problem set), but not all walks might reach the stationary distribution if started from an arbitrary vertex. For example, if $G$ has many connected components, then a random walk will stay in it’s own connected component. Also, $G$ is bipartite, then a walk will oscillate between the two sides. We will show in the analysis below that these are essentially the only two obstacles to reaching a stationary distribution.
2.1 Random walks on regular graphs

One problem in applying some of the theory from previous lectures is that the matrix $M$ is not symmetric. However, if $G$ is $d$-regular (each vertex has degree $d$), then the matrix $M$ becomes

$$ M = AD^{-1} = \frac{1}{d} \cdot A, $$

and is a symmetric matrix. We will see later how to extend the ideas to general graphs. We will also assume that the graph $G$ is connected, as otherwise we can analyze the walk in each connected component separately. Then the eigenvalues for the matrix $M$ are $\mu_1/d, \ldots, \mu_n/d$ and the eigenvectors are the same as those for the matrix $A$.

**Exercise 2.2** Suppose $G$ is $d$-regular. Then $x = (1/n, \ldots, 1/n)^T$ is a stationary distribution for the simple random walk on $G$.

Let $\mu = \max_{i=2, \ldots, n} |\mu_i| = \max\{\mu_2, -\mu_n\}$. We will show that the distribution of the random walk converges to the stationary distribution as long as $\mu < d$. Recall that if $G$ is connected iff $\mu_2 < d$ and non-bipartite iff $\mu_n > -d$.

**Lemma 2.3** Let $G$ be a $d$-regular graph and let $\mu = \max\{\mu_2, -\mu_n\}$. Then, after $t$ steps of a simple random walk on $G$ started at an arbitrary vertex $i_0$, we have that

$$ \forall i \in V, \quad |p^{(t)}(i) - 1/n| \leq \left(\frac{\mu}{d}\right)^t. $$

**Proof:** Let $M = \frac{1}{d} A$ such that $p^{(t)} = M^t p^{(0)}$. Let $u_1, \ldots, u_n$ be the orthonormal eigenbasis of $M$ such that we can write $p^{(0)} = \sum_{i=1}^n \alpha_i u_i$.

$$ p^{(t)} = M^t p^{(0)} $$
$$ = M^t \left( \sum_i \alpha_i u_i \right) $$
$$ = \sum_i \alpha_i (\mu_i/d)^t u_i \quad (\text{Since } M^t u_i = (\mu_i/d)^t u_i) $$

What is $u_1$? Setting $u_1 = c(1/n, \ldots, 1/n)^T$, $\langle u_1, u_1 \rangle = \sum_i c^2(1/n)^2 = 1$ gives us $u_1 = (1/\sqrt{n}, \ldots, 1/\sqrt{n})^T$.

What is $\alpha_1$? Since $p^{(0)} = \sum_i \alpha_i u_i$ and $u_1, \ldots, u_n$ form an orthonormal basis,

$$ \alpha_1 = \langle p^{(0)}, u_1 \rangle = \frac{1}{\sqrt{n}} \sum_i p_i^{(0)} = \frac{1}{\sqrt{n}} $$

So we have

$$ p^{(t)} = M^t p^{(0)} $$
$$ = \frac{1}{\sqrt{n}} \left( \frac{d}{d} \right)^t \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix} + \alpha_2 \left( \frac{\mu_2}{d} \right)^t u_2 + \cdots + \alpha_n \left( \frac{\mu_n}{d} \right)^t u_n $$

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Thus we have
\[ p^{(t)} = \left( \frac{1}{n} \right)^t = \alpha_2 \left( \frac{\mu_2}{d} \right)^t u_2 + \ldots + \alpha_n \left( \frac{\mu_n}{d} \right)^t u_n \]

Let \( e \) denote the “error vector” \( \alpha_2 \left( \frac{\mu_2}{d} \right)^t u_2 + \ldots + \alpha_n \left( \frac{\mu_n}{d} \right)^t u_n \). We need to show that for each \( i \in V, |e(i)| \leq (\mu/d)^t \). The following claim finishes the proof.

**Claim 2.4** \( \|e\| = \sqrt{\sum_i |e(i)|^2} \leq \frac{\mu}{d} \).

**Proof:**
\[ \|e\|^2 = \langle e, e \rangle = \sum_{i=2}^{n} \alpha_i^2 \left( \frac{\mu_i}{d} \right)^2 \leq \left( \sum_{i=2}^{n} \alpha_i^2 \right) \cdot \left( \frac{\mu}{d} \right)^2 \leq \left( \frac{\mu}{d} \right)^2 . \]

Here the last inequality follows from the fact that \( \langle p^{(0)}, p^{(0)} \rangle = \sum_{i=1}^{n} \alpha_i^2 = 1 \).

### 2.2 Random walks on general undirected graphs

For general graphs \( G \), we have \( M = AD^{-1} \) as defined above. Let \( D^{-\frac{1}{2}} \) be the diagonal matrix with entries \( (D^{-\frac{1}{2}})_{ii} = \frac{1}{\sqrt{\text{deg}(i)}} \). The analysis for random walks is very similar to the above, but we use the matrix \( D^{-1/2}AD^{-1/2} \) which is similar to the matrix \( M = AD^{-1} \).

**Exercise 2.5** Show that \( AD^{-1} = M \sim D^{-\frac{1}{2}}AD^{-\frac{1}{2}} \).

Thus, the eigenvalues of the two matrices are the same there is an isomorphic between their eigenspaces for each eigenvalue. However, the matrix \( D^{-\frac{1}{2}}AD^{-\frac{1}{2}} \) is symmetric and has real eigenvalues and an orthonormal basis of real eigenvectors.

Random walks on a connected undirected graph can be analyzed by expressing the initial distribution in terms of the eigenvectors of the matrix \( D^{-\frac{1}{2}}AD^{-\frac{1}{2}} \). We will leave the details to the problem set.