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1 Coloring a 3-Colorable Graph (contd.)

Continuing calculation form previous lecture

$$\begin{aligned}\mathbb{E}\left[|S'|\right] &\geq n \;.\; F'(l) - \frac{nD}{2} \;.\; F(2l) \\ &= n \left(F(l) - \frac{nD}{2} \;.\; F(2l) \right)\end{aligned}$$

Claim 1.1 *For $l \geq 1$, we have:*

$$\frac{1}{\sqrt{2\Pi}} \frac{e^{-l^2/2}}{2l} \leq F(l) \leq \frac{1}{\sqrt{2\Pi}} \frac{e^{-l^2/2}}{l}$$

If there is such a bound, we have:

$$\begin{aligned}\mathbb{E}\left[|S'|\right] &\geq \frac{n}{\sqrt{2\Pi}} \left(\frac{e^{-l^2/2}}{2l} - \frac{D}{2} \cdot \frac{e^{-(2l)^2/2}}{2l} \right) \\ &= \frac{n}{\sqrt{2\Pi}} \frac{e^{-l^2/2}}{2l} \left(1 - \frac{D}{2} \;.\; e^{-3l^2/2} \right)\end{aligned}$$

Where D is the average degree.

We can choose l such that: $e^{-3l^2/2} = \frac{1}{D}$. So we set $l = \sqrt{\frac{2}{3} \ln D}$.

$$\Rightarrow \mathbb{E}\left[|S'|\right] \geq \frac{n}{\sqrt{2\Pi}} \cdot \frac{1}{D^{1/3}} \cdot \frac{1}{2\sqrt{\frac{2}{3}\ln D}}$$

So $k = O(D^{1/3}\sqrt{\ln D})$. Give that D is at most n, we have $k = O(n^{1/3}\sqrt{\ln n})$.

So we have a coloring with $O(n^{1/3}\sqrt{\ln n} \;.\; \ln n) = O(n^{1/3}(\ln n)^{3/2})$ colors.

Now let's prove claim 1.1.

Proof of claim: [CMM06]

The idea is simple: we just do integration by parts.

$$
\begin{aligned}
\sqrt{2\Pi}F(l) &= \int_l^\infty e^{-y^2/2}\,\mathrm{d}y \\
&= \int_l^\infty \frac{1}{y}\,.\,ye^{-y^2/2}\,\mathrm{d}y && (l > 0,\text{ so } y > 0) \\
&= \frac{1}{y}(-e^{y^2/2})\Big|_l^\infty - \int_l^\infty (\frac{1}{y^2})\,.\,(e^{-y^2/2})\,\mathrm{d}y \\
&= \frac{e^{-l^2/2}}{l} - \int_l^\infty \frac{e^{-y^2/2}}{y^2}\,\mathrm{d}y \\
&\geq \frac{e^{-l^2/2}}{l} - \sqrt{2\Pi}\,F(l) && (\text{since } y > 0)
\end{aligned}
$$

This was the lower bound. Upper bound is obvious, since a positive value is subtracted. □

2 Spectral Graph Theory

Consider $M \in R^{n\times n}$, which is symmetric. We can write M as:

$$
\begin{aligned}
&M = \sum \mu_i \mathbf{v}_i \mathbf{v}_i^T \\
&\mathbf{v}_1, \dots, \mathbf{v}_n \in R^n,\ \text{ orthonormal} \\
&\mu_1, \dots, \mu_n \in R,\ \text{ eigen values}
\end{aligned}
$$

$$
\forall j\ \ M\mathbf{v}_j = \mu_j \mathbf{v}_j\ \ \text{, or equally: } \mathbf{v}_j^T M \mathbf{v}_j = \mu_j
$$

From now on, we assume $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

$$
\begin{aligned}
\mu_1 &= \max_{\mathbf{x}\in R^n} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \\
\mu_2 &= \max_{\mathbf{x}\perp \mathbf{v}_1} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T} \\
&\vdots \\
\mu_n &= \min_{\mathbf{x}\in R^n} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}
\end{aligned}
$$

Side note: The full theorem is called Courant-Fisher theorem. It is recommended that you look it up.

Graphs and Matrices

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{o.w.} \end{cases}$$

A is symmetric, so we can write it as $\sum_{i=1}^{n} \mu_i v_i v_i^T$.
Let's look at an interesting property of A:

$$\text{average degree} \leq \mu_1 \leq \text{maximum degree}$$

Proof: First, we prove the lower-bound.

$$\mu_1 = \max_{\mathbf{x} \in R^n} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geq \frac{\mathbf{1}^T A \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{\sum_{i,j} A_{ij}}{n} = \frac{2m}{n} = D$$

Now, we show that the upper-bound holds.
Suppose $\mathbf{v}$is an eigenvector with eigenvalue μ_1. We have $A\mathbf{v} = \mu_1 \mathbf{v}$, so:

$$\begin{aligned} \forall i \quad & \sum_j A_{ij} \mathbf{v}_j = \mu_1 \mathbf{v}_i \\ \Rightarrow \ & |\mathbf{v}_i| \leq \frac{1}{\mu_i} \left| \sum_j A_{ij} \mathbf{v}_j \right| \\ & \leq \frac{1}{\mu_i} \left(\sum_j |A_{ij}| \right) \max_j |\mathbf{v}_j| \end{aligned}$$

By defining $i^* = \text{argmax}_i \, |\mathbf{v}_i|$, we have:

$$\begin{aligned} & |\mathbf{v}_{i^*}| \leq \frac{1}{\mu_1} \left(|A_{ij}| \right) |\mathbf{v}_{i^*}| \\ \Rightarrow \ & \mu_1 \leq \sum_j |A_{i^* j}| \leq D_{\max} \end{aligned}$$

■

Back to the Coloring Problem

Minimum number of colors needed to color G is called the *chromatic number* of G, shown as $\mathcal{X}(G)$.
We are going to show that $\mathcal{X}(G) \leq \lfloor \mu_1 \rfloor + 1$.

Proof: We proceed with induction on $n = |V|$
For $n = 1$, $A = [0]$, which is 1-colorable.

Induction step: consider a general graph G.

$$\begin{aligned} \exists i \in V \quad & \deg(i) \leq \mu_1 \\ \text{or equivalently} \quad & \deg(i) \leq \lfloor \mu_1 \rfloor \end{aligned}$$

Consider graph $G \setminus \{i\}$ and color it with $\lfloor \mu_1' \rfloor + 1$ colors.

Claim 2.1 $\mu_1' \leq \mu_1$
We will prove this claim later.

The vertex that was left out, i, has degree of at most $\lfloor \mu_1 \rfloor$, because we picked it to be so. So there is one color left for i. ■

Proof: Here we prove claim 2.1. Let A' be adjacency matrix of $G \setminus \{i\}$.

$$\mu_1' = \max_{\mathbf{x} \in R^{n-1}} \frac{\mathbf{x}^T A' \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

For every $\mathbf{x} \in R^{n-1}$, if we define a $\mathbf{z} \in R^n$ as $\binom{\mathbf{x}}{0}$, we can check that we have $\mathbf{z}^T \mathbf{z} = \mathbf{x}^T \mathbf{x}$ and

$$\begin{aligned} \mathbf{x}^T A' \mathbf{x} &= \sum_{k,l \neq i} A'_{kl} \mathbf{x}_k \mathbf{x}_l \\ &= \sum_{k,l} A_{kl} \mathbf{z}_k \mathbf{z}_l \\ &= \mathbf{z}^T A \mathbf{z} \end{aligned}$$

So we will have:

$$\mu_1' = \max_{\mathbf{z} \in R^{n-1} \times \{0\}} \frac{\mathbf{z}^T A \mathbf{z}}{\mathbf{z}^T \mathbf{z}} \leq \max_{\mathbf{z} \in R^n} \frac{\mathbf{z}^T A \mathbf{z}}{\mathbf{z}^T \mathbf{z}} = \mu_1$$

■

In summary, we showed that $\mathcal{X}(G) \leq \lfloor \mu_1 \rfloor + 1$. This means that G has an independent set of size at least $\frac{n}{\lfloor \mu_1 \rfloor + 1}$.

Laplacian Matrix

We define the Laplacian matrix as:

$$L = D - A$$

where A is the adjacency matrix, and D a diagonal matrix with $D_{ii} = deg(i)$.

Normalized Laplacian is defined as:

$$N = D^{-1/2} L D^{-1/2} = I - D^{-1/2} A D^{-1/2}$$

Using the definitions, let us take a look at what $\mathbf{x}^T L \mathbf{x}$ corresponds to:

$$\begin{aligned}
\mathbf{x}^T L \mathbf{x} &= \mathbf{x}^T (D - A)\mathbf{x} \\
&= \mathbf{x}^T D \mathbf{x} - \mathbf{x}^T A \mathbf{x} \\
&= \sum_i deg(i) x_i^2 - \sum_{(i,j)\in E} 2x_i x_j \\
&= \sum_i \sum_{(i,j)\in E} x_i - \sum_{(i,j)\in E} 2x_i x_j \\
&= \sum_{(i,j)\in E} \left(x_i^2 + x_j^2 - 2x_i x_j\right) \\
&= \sum_{(i,j)\in E} (x_i - x_j)^2
\end{aligned}$$

Now consider the d-regular graph G. We have:

$$L = dI - A$$

$$N = (I - \frac{1}{d} A)$$

If $\mathbf{v}$ is an eigenvector of A, we have $A\mathbf{v} = \mu \mathbf{v}$. Thus, $(dI - A)\mathbf{v} = (d - \mu)\mathbf{v}$.
Let $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be eigen values of N. We have $\lambda_i = 1 - \frac{\mu_i}{d}$.

Exercise 2.2 *Prove* $|\mu_n| \leq d$*.*

References

[C96] F. Chung, "Spectral Graph Theory", *American Mathematical Society*, 1996.

[S12] D. Speilman, "Lecture Notes on Spectral Graph Theory", *Yale University*, 2012.