1. Gaussian Random Variables. \[5+5+5\]

Prove the following very useful facts about Gaussian random variables:

(a) Let \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \) be two vectors. Let \( \mathbf{g} \in \mathbb{R}^n \) be a random vector such that each coordinate \( g_i \) of \( \mathbf{g} \) is distributed as a Gaussian random variable with mean 0 and variance 1, and any two coordinates \( g_i, g_j \) (for \( i \neq j \)) are independent. Then show that

\[
\mathbb{E} \left[ \langle \mathbf{u}, \mathbf{g} \rangle \cdot \langle \mathbf{v}, \mathbf{g} \rangle \right] = \langle \mathbf{u}, \mathbf{v} \rangle .
\]

(b) Let \( \mathbf{g} \) be a Gaussian random variable with mean 0 and variance 1. Show that for any \( t \in \mathbb{R} \), we have

\[
\mathbb{E} \left[ e^{t \mathbf{g}} \right] = e^{t^2/2}.
\]

Comparing coefficients of \( t^{2k} \) on both sides, use this to show that for any \( k \in \mathbb{N} \),

\[
\mathbb{E} \left[ \mathbf{g}^{2k} \right] = \frac{(2k)!}{2^k \cdot k!} .
\]

(c) Let \( g_1, g_2, g_3 \) and \( g_4 \) be (not necessarily independent) Gaussian random variables with mean 0. Additionally, assume that for all coefficients \( a_1, \ldots, a_4 \in \mathbb{R} \), the linear combination \( a_1 g_1 + \cdots + a_4 g_4 \) is also a Gaussian random variable (note that this is not always true if \( g_1, \ldots, g_4 \) are not independent, but here we are restricting ourselves to \( g_1, \ldots, g_4 \) which satisfy this assumption).

Consider the function \( \mathbb{E}_{g_1,g_2,g_3,g_4} \left[ e^{t_1 g_1 + t_2 g_2 + t_3 g_3 + t_4 g_4} \right] \) in the variables \( t_1, t_2, t_3, t_4 \) and use it to show that

\[
\mathbb{E} \left[ g_1 g_2 g_3 g_4 \right] = \mathbb{E} \left[ g_1 g_2 \right] \cdot \mathbb{E} \left[ g_3 g_4 \right] + \mathbb{E} \left[ g_1 g_3 \right] \cdot \mathbb{E} \left[ g_2 g_4 \right] + \mathbb{E} \left[ g_1 g_4 \right] \cdot \mathbb{E} \left[ g_2 g_3 \right] .
\]

This shows that for any four Gaussian random variables, the expectation of their product can be expressed in terms of their pairwise correlations! This is a special case of what is known as Wick’s theorem, which can also be proved by the above method.
2. Supremum of Gaussians. [5+5]

(a) Let $g \sim N(0, 1)$ be a Gaussian random variable with mean 0 and variance 1. Show that for $t \geq 1$

$$\mathbb{P}[g \geq t] = \int_t^\infty \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \, dx \leq e^{-t^2/2}.$$ 

(b) Let $g_1, \ldots, g_n \sim N(0, 1)$ be independent Gaussian random variables. Show that

$$\mathbb{E} \left[ \max_{i \in [n]} |g_i| \right] \leq 4\sqrt{\ln n}.$$ 

You may use the fact that for a non-negative random variable $Z$, the expectation can be computed as $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}[Z \geq t] \, dt$. 

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