Mathematical Toolkit **Autumn 2021**

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1 Self-adjoint transformations

Definition 1.1 *A linear transformation $\varphi : V \to V$ is called* self-adjoint *if $\varphi = \varphi^*$. Note that such a transformation necessarily needs to map v to itself, and is thus a linear operator.*

Example 1.2 *The transformation represented by matrix $A \in \mathbb{C}^{n\times n}$ is self-adjoint if $A = \overline{A^T}$. Such matrices are called Hermitian matrices.*

Proposition 1.3 *Let V be an inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then*

- *All eigenvalues of φ are real.*
- *If $\{w_1, \ldots, w_n\}$ are eigenvectors corresposnding to distinct eigenvalues then they are mutually orthogonal.*

Proof: The first property can be observed by noting that if $v \in V \setminus \{0_V\}$ is an eigenvector with eigenvalue λ, then

$$\lambda \cdot \langle v, v\rangle \;=\; \langle v, \lambda \cdot v\rangle \;=\; \langle v, \varphi(v)\rangle \;=\; \langle \varphi^*(v), v\rangle \;=\; \langle \varphi(v), v\rangle \;=\; \overline{\lambda} \cdot \langle v, v\rangle \;.$$

Since $\langle v, v\rangle \neq 0$, we must have $\lambda = \overline{\lambda}$ which implies that $\lambda \in \mathbb{R}$. For the second part, observe that if $i \neq j$, then we have

$$\lambda_j \cdot \langle w_i, w_j\rangle \;=\; \langle w_i, \varphi(w_j)\rangle \;=\; \langle \varphi^*(w_i), w_j\rangle \;=\; \langle \varphi(w_i), w_j\rangle \;=\; \overline{\lambda_i} \cdot \langle w_i, w_j\rangle \;.$$

Since eigenvalues are real, we get $(\lambda_i - \lambda_j) \cdot \langle w_i, w_j\rangle = 0$, which implies $\langle w_i, w_j\rangle = 0$ using $\lambda_i \neq \lambda_j$. ■

2 The Real Spectral Theorem

In this lecture, we will prove the "real spectral theorem" for self-adjoint operators $\varphi : V \to V$ (so named because the eigenvalues of a self-adjoint operator are real, not because other spectral theorems are fake!) We will show that any such operator is not only diagonalizable (has a basis of eigenvectors) but is in fact *orthogonally diagonalizable* i.e., has an *orthonormal* basis of eigenvectors. This gives a very convenient way of thinking about the action of such operators. In particular, let $\dim(V) = n$ and $\{w_1, \ldots, w_n\}$ form an orthonormal basis of eigenvectors for φ, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any vector v expressible in this basis as (say) $v = \sum_{i=1}^{n} c_i \cdot w_i$, we can think of the action of φ as

$$\varphi(v) \;=\; \varphi\left(\sum_{i=1}^{n} c_i \cdot v_i\right) \;=\; \sum_{i=1}^{n} c_i \cdot \lambda_i \cdot w_i \,.$$

Of course, we can also think of the action of φ in this way as long as $w_1, \ldots, w_n$ form a basis (not necessarily orthonormal). However, this view is particularly useful when they form an orthonormal basis. As we will later see, this also provides the "right" basis to think about many matrices, such as the adjacency matrices of graphs (where such decompositions are the subject of spectral graph theory). To prove the spectral theorem, We will need the following statement (which we'll prove later).

Proposition 2.1 *Let V be a finite-dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ has at least one eigenvalue.*

Using the above proposition, we will prove the spectral theorem below for finite dimensional vector spaces. The proof below can also be made to work for Hilbert spaces (using the axiom of choice). The above proposition, which gives the existence of an eigenvalue is often proved differently for finite and infinite-dimensional spaces, and the proof for infinite-dimensional Hilbert spaces requires additional conditions on the operator φ. We first prove the spectral theorem assuming the above proposition.

Proposition 2.2 (Real spectral theorem) *Let V be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ is orthogonally diagonalizable.*

Proof: By induction on the dimension of V. Let $\dim(V) = 1$. Then by the previous proposition φ has at least one eigenvalue, and hence at least one eigenvector, say w. Then $w / \|w\|$ is a unit vector which forms a basis for V.

Let $\dim(V) = k + 1$. Again, by the previous proposition φ has at least one eigenvector, say w. Let $W = \text{Span}(\{w\})$ and let $W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0\}$. Check the following:

- $W^{\perp}$ is a subspace of V.

- $\dim(W^\perp) = k$.
- $W^\perp$ is invariant under φ i.e., $\forall v \in W^\perp$, $\varphi(v) \in W^\perp$.

Thus, we can consider the operator $\varphi' : W^\perp \to W^\perp$ defined as

$$\varphi'(v) \;:=\; \varphi(v) \qquad \forall v \in W^\perp .$$

Then, φ' is a self-adjoint (check!) operator defined on the k-dimensional space $W^\perp$. By the induction hypothesis, there exists an orthonormal basis $\{w_1, \ldots, w_k\}$ for $W^\perp$ such that each w_i is an eigenvector of φ. Thus $\left\{w_1, \ldots, w_k, \frac{w}{\|w\|}\right\}$ is an orthonormal basis for V, comprising of eigenvectors of φ. ∎

3 Existence of eigenvalues

We now prove Proposition 2.1, which shows that a self-adjoint operator must have at least one eigenvalue. Let us assume for now that V is an inner product space over $\mathbb{C}$. As was observed in class, in this case we don't need self-adjointness to guarantee an eigenvalue. We thus prove the following more general result

Proposition 3.1 *Let V be a finite dimensional vector space over $\mathbb{C}$ and let $\varphi : V \to V$ be a linear operator. Then φ has at least one eigenvalue.*

Proof: Let $\dim(V) = n$. Let $v \in V \setminus 0_V$ be any non-zero vector. Consider the set of $n+1$ vectors $\{v, \varphi(v), \ldots, \varphi^n(v)\}$. Since the dimension of V is n, there must exists $c_0, \ldots, c_n \in \mathbb{C}$ such that

$$c_0 \cdot v + c_1 \cdot \varphi(v) + \cdots + c_n \varphi^n(v) \;=\; 0_V .$$

We assume above that $c_n \neq 0$, otherwise we can only consider the sum to the largest i such that $c_i \neq 0$. Let $P(x)$ denote the polynomial $c_0 + c_1 x + \cdots + c_n x^n$. Then the above can be written as $(P(\varphi))(v) = 0$, where $P(\varphi) : V \to V$ is a linear operator defined as

$$P(\varphi) \;:=\; c_0 \cdot \mathsf{id} + c_1 \cdot \varphi + \cdots + c_n \varphi^n ,$$

with id used to denote the identity operator. Since P is a degree-n polynomial over $\mathbb{C}$, it can be factored into n linear factors, and we can write $P(x) = c_n \prod_{i=1}^{n} (x - \lambda_i)$ for $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. This means that we can write

$$P(\varphi) \;=\; c_n(\varphi - \lambda_n \cdot \mathsf{id}) \cdots (\varphi - \lambda_1 \cdot \mathsf{id}) .$$

Let $w_0 = v$ and define $w_i = \varphi(w_{i-1}) - \lambda_i \cdot w_{i-1}$ for $i \in [n]$. Note that $w_0 = v \neq 0_V$ and $w_n = P(\varphi)(v) = 0_V$. Let i^* denote the largest index i such that $w_i \neq 0_V$. Then, we have

$$0_V \;=\; w_{i^*+1} \;=\; \varphi(w_{i^*}) - \lambda_{i^*+1} \cdot w_{i^*} .$$

This implies that w_{i^*} is an eigenvector with eigenvalue λ_{i^*+1}. ∎

To prove Proposition 2.1 using this, we note that $\varphi = \varphi^*$ implies the eigenvalue found by the above proposition must are real.

Exercise 3.2 *Use the fact that the eigenvalues of a self-adjoint operator are real to prove Proposition 2.1 even when V is an inner product space over $\mathbb{R}$.*

Hint: Define a "complex extension" $V' = \{u + iv \mid u, v \in V\}$, which is a vector space over $\mathbb{C}$ with the scalar multiplication rule

$$(a + ib) \cdot (u + iv) \; = \; (a \cdot u - b \cdot v) + i(a \cdot v + b \cdot u)\,.$$

Also, extend φ to φ' defined as $\varphi' : V' \to V'$ with $\varphi'(u + iv) = \varphi(u) + i\varphi(v)$. Then, φ' has at least one (possibly complex) eigenvalue by the previous result. Can you use it to deduce the existence of a real eigenvalue for φ.