Errata of “Lower and Upper Bounds on the Generalization of Stochastic Exponentially Concave Optimization”

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Abstract

We fix two typos in the statement of Theorem 4, and an error in Theorem 8. To be more clear, we rewrite the proof of the lower bound.

1 Statement of Theorem 4

\[ |X_i^2| < R \rightarrow |X_i| \leq R \]
\[ \sqrt{2} R \sqrt{\log \frac{2t+1}{\delta^2}} \rightarrow R \sqrt{\log \frac{2t+1}{\delta^2}} \]

2 Proof of the Lower Bound

We now show that for square loss, which is a special case of exponentially concave functions, the minimax risk is \( O(d/T) \). As a result, the online Newton step algorithm achieves the almost optimal excess risk bound. The proof of the lower bound is built upon the distance-based Fano inequality (Duchi and Wainwright, 2013).

Let \( \mathcal{P} \) be a family of distributions on a sample space \( \mathcal{X} \), and let \( \theta : \mathcal{P} \mapsto \Theta \) be a function mapping \( \mathcal{P} \) to some parameter space \( \Theta \). Given a set of \( n \) samples \( X^n = \{X_1, \ldots, X_n\} \) drawn i.i.d. from a distribution \( P \in \mathcal{P} \), let \( \hat{\theta}(X^n) \) be a measurable function of \( X^n \), which is an estimate of the unknown quantity \( \theta(P) \). Then, the minimax risk for the family \( \mathcal{P} \) is given by

\[
\mathcal{M}_n (\theta(P), \Phi \circ \rho) = \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} E_P \left[ \Phi \left( \rho \left( \hat{\theta}(X^n), \theta(P) \right) \right) \right]
\]

where \( \rho : \Theta \times \Theta \mapsto \mathbb{R} \) is a (semi)-metric on the parameter space, and \( \Phi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is a nondecreasing loss function. Our analysis is based on the following result from Duchi and Wainwright (2013).

**Lemma 1** (Corollary 2 of Duchi and Wainwright (2013)). Let’s consider a discrete set \( \mathcal{V} \) and each element \( v \in \mathcal{V} \) leads to a vector \( \theta_v \in \Theta \) that results in a distribution \( P \in \mathcal{P} \). Given a function \( \rho_\mathcal{V} : \mathcal{V} \times \mathcal{V} \mapsto \mathbb{R} \) and a scalar \( t \), we define the separation function

\[
\delta(t) := \sup \{ \delta \| \rho(\theta_v, \theta_w) \geq \delta \text{ for all } v, w \in \mathcal{V} \text{ such that } \rho_\mathcal{V}(v, w) > t \}.
\]
We assume the canonical estimation setting: nature chooses a vector $V \in \mathcal{V}$ uniformly at random, and conditioned on this choice $V = v$, a sample $X^n$ of size $n$ is drawn i.i.d. from the distribution $P \in \mathcal{P}$ with parameter $\theta_v$. Then, we have

$$\mathfrak{M}_n(\theta(P), \Phi \circ \rho) \geq \Phi \left( \frac{\delta(t)}{2} \right) \left( 1 - \frac{I(X^n; V) + \log 2}{\log |\mathcal{V}| - \log N^\text{max}_t} \right), \quad \forall t$$

where $N^\text{max}_t = \max_{v \in \mathcal{V}} \{|\mathcal{V}'| \rho \mathcal{V}(v, \mathcal{V}') \leq t\}$.

In our case, we are interested the generalization error bound $\mathcal{L}(\hat{w}) - \mathcal{L}(w_*)$. For square loss, the stochastic optimization problem is given by

$$\min_{w \in \mathcal{W}} \mathcal{L}(w) = E \left[ (Y - X^\top w)^2 \right]$$

where $X$ is sampled from some underlying distribution $P_X$, and given $X = x$ the response $Y$ is sampled from an Gaussian distribution $\mathcal{N}(x^\top w_*, 1)$, where $w_* \in \mathbb{R}^d$ is the parameter vector. Furthermore, we assume $w_* \in \mathcal{W}$. Then, it is easy to verify that the excess risk of a solution $\hat{w}$ is

$$\mathcal{L}(\hat{w}) - \mathcal{L}(w_*) = E \left[ (X^\top \hat{w} - X^\top w_*)^2 \right] = (\hat{w} - w_*)^\top E[X X^\top] (\hat{w} - w_*) = \|\hat{w} - w_*\|^2_C$$

where we define $C = E[X X^\top]$. Then, the semi-metric is naturally defined as

$$\rho(w, w') = \|w - w\|_C$$

and $\Phi(z) = z^2$. Let $\mathcal{P}_{X,Y}$ be a family of joint distributions of $X$ and $Y$. Using these notations, the minimax risk for the generalization error bound becomes

$$\mathfrak{M}_T (\theta(\mathcal{P}_{X,Y}), \Phi \circ \rho) = \inf_{\hat{w}} \sup_{P \in \mathcal{P}_{X,Y}} E_P \left[ \|\hat{w}((X, Y)^T) - w(P)\|_C \right]$$

where $w(P)$ is used to represent the parameter vector for distribution $P$, $(X, Y)^T = \{(X_1, Y_1), \ldots, (X_T, Y_T)\}$ are $T$ samples drawn from $P$ and $\hat{w}(\cdot)$ is a measurable function of $(X, Y)^T$.

To utilize Lemma 1, we introduce a discrete set $\mathcal{V} = \{v \in \{-1, 0, 1\}^d \mid \|v\|_0 = c_1 d\}$ for some constant $c_1 < 1$, define $w_v = \varepsilon v$ for $\varepsilon > 0$, and assume $w_* \in \{\varepsilon v : v \in \mathcal{V}\} \subseteq \mathcal{W}$. In our analysis, we set $t = c_2 d$ with $c_2 < c_1$, and define $\rho_\mathcal{V}(v, w) = \|v - w\|_0$. Then, we lower bound the separation function $\delta(\cdot)$ by

$$\delta(c_2 d) = \sup \{\varepsilon \|v - w\|_C \geq \delta \text{ for all } v, w \in \mathcal{V} \text{ such that } \|v - w\|_0 > c_2 d\}$$

$$= \min \{\varepsilon \|v - w\|_C \mid \text{ for all } v, w \in \mathcal{V} \text{ such that } \|v - w\|_0 > c_2 d\}$$

$$\geq \min \left\{ \varepsilon \|z\|_C \mid \text{ for all } z \in \{-2, -1, 0, +1, +2\}^d \text{ such that } c_2 d < \|z\|_0 \leq 2c_1 d \right\}$$

$$\geq \varepsilon \sqrt{c_2 d} \min_{z \geq \mu} \{\|z\|_C \mid \text{ for all } \|z\|_2 \geq 1, \|z\|_0 \leq 2c_1 d \}$$

Using Lemma 1, we have

$$\mathfrak{M}_T (\theta(\mathcal{P}_{X,Y}), \Phi \circ \rho) > c_2 d \varepsilon^2 \mu^2 \left( 1 - \frac{I(V; (X, Y)^T) + \log 2}{\log |\mathcal{V}| - \log N^\text{max}_t} \right).$$
In addition, we have

\[ I(V; (X, Y)^T) = TI(V; (X, Y)) \]

and

\[ I(V; (X, Y)) = H(X, Y) - H(X, Y|V) \]

\[ = H(X) + H(Y|X) - H(X|V) - H(Y|X, V) = H(Y|X) - H(Y|X, V) \]

\[ \leq E\left[ \frac{1}{|V|^2} \sum_{w \in V} \sum_{v \in V} D_{kl}\left( N(\varepsilon X^T w, 1)|N(\varepsilon X^T v, 1) \right) \right] \]

\[ = \frac{\varepsilon^2}{2} E\left[ (V - W)^T XX^T (V - W) \right] = \frac{\varepsilon^2}{2} E\left[ \text{tr}\left( XX^T (V - W)(V - W)^T \right) \right] = \varepsilon^2 c_1 \text{tr}(C) \]

where \( V \) and \( W \) are two independent random variables that are uniformly distributed on \( V \), which implies \( E[VV^T] = E[WW^T] = c_1 I \) and \( E[VW^T] = 0 \). Furthermore, it is easy to verify

\[ \log |V| - \log N_t^{\max} \geq c_3 d \]

for some constant \( c_3 > 0 \) when \( d \) is large enough and \( c_2 \) is small enough. Combining the above result, we have

\[ \mathfrak{M}_T (\theta(P_{X,Y}), \Phi \circ \rho) \geq c_2 d \varepsilon^2 \mu^2 \left( 1 - \frac{T \varepsilon^2 c_1 \text{tr}(C)}{c_3 d} \right) = \frac{c_2 c_3 d}{4Tc_1} \cdot \frac{d \mu^2}{\text{tr}(C)} \]

where we choose \( \varepsilon^2 = \frac{c d}{27 c_1 \text{tr}(C)} \).

To show the minimax risk is of \( O(d/T) \), we need to construct a matrix \( C \) such that \( \text{tr}(C) = O(d) \) and \( \mu^2 \) is a sufficiently large constant. Furthermore, to ensure the optimization problem is exponential concave instead of strongly convex, \( C \) should be singular. The existence of such a matrix is guaranteed by the following theorem.

**Theorem 1.** When \( c_1 \) is smaller enough, there exists a singular matrix \( C \) such that \( \text{tr}(C) = d \) and \( \mu^2 \geq 1/2 \).

**Proof.** We prove this theorem by utilizing the uniform uncertainty principle of subgaussian matrices (Mendelson et al., 2008). Let \( R \in \mathbb{R}^{d \times k} \) be a random matrix with \( R_{ij} \) sampled uniformly from \{\pm 1\}. Following Corollary 3.3 of Mendelson et al. (2008), with a probability at least \( 1 - \exp(-ck) \)

\[ z^T \frac{RR^T}{k} z \geq \frac{1}{2} \|z\|_2^2 \text{ for all } \|z\|_0 \leq \frac{k}{c' \log d} \]

for some constant \( c, c' > 0 \). By choosing \( C = \frac{RR^T}{k} \) and \( k = 2c' c_1 d \log d \), with a probability at least \( 1 - \exp(-2cc' c_1 d \log d) \), we have

\[ \mu = \min \{ \|z\|_C \mid \text{ for all } \|z\|_2 \geq 1, \|z\|_0 \leq 2c_1 d \} \geq \frac{\sqrt{2}}{2}. \]

Since the success probability \( 1 - \exp(-2cc' c_1 d \log d) \) is strictly greater than 0, there must exist such a matrix \( C \). From our construction of \( R \), it is easy to verify \( \text{tr}(C) = d \) and when \( c_1 < 1/(2c' \log d) \), we have \( k < d \) and thus \( C \) is singular. \( \square \)
References
