Optimization

Problem set 1

Due Monday, April 12th

1. Consider the closed convex set $B_1 = \{ x \in \mathbb{R}^n | \|x\|_1 = \sum_i |x_i| \leq 1 \}$. This is the unit ball of the $\ell_1$ norm.

   (a) Show that $B_1$ is a polyhedron by explicitly expressing it as an intersection of halfspaces. How many halfspaces (“facets”) are required in order to express $B_1$?

   (b) Explicitly express $B_1$ as a convex hull of a finite number of points. How many points (“vertices”) are required in this characterization?

   (c) Contrast this with the $\ell_\infty$ unit ball, $B_\infty = \{ x \in \mathbb{R}^n | \|x\|_\infty \leq 1 \}$. How many halfspaces are required in order to express $B_\infty$ as an intersection of halfspaces? How many points are required in order to express $B_\infty$ as a convex hull?

   (d) For each point $\hat{x}$ on the boundary of $B_1$, identify the set of all supporting hyperplanes of $B_1$ at $\hat{x}$ explicitly. For each such $\hat{x}$, what is the dimensionality of this set?

2. Consider a polyhedron $C = \text{conv} \{ v_1, \ldots, v_k \} \subset \mathbb{R}^n$ and a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

   (a) Prove that a maximum of $f$ over $C$ is achieved at one of the vertices $v_i$. (Hint: assume the statement is false and use Jensen’s inequality). Is it possible that the maximum is also achieved at an interior point?

   (A generalization of the above is that a maximum of a function over a closed and bounded convex set is achieved at an extreme point, i.e. a point which is not a convex combination of other points in the set).

   (b) Use the above to conclude that the minimum of a linear objective over the polyhedron $C$ is always achieved at one of the vertices $v_i$. 

3. In this problem we will define strong convexity more generally then it is defined by Boyd and Vandenberghe (Section 9.1.2). In particular, we will consider a definition that is valid also for non-differentiable functions.

**Definition:** A function $f : \mathbb{R}^n \to \mathbb{R}$ is $m$-strongly convex if for every $x, y \in \mathbb{R}^n$ and every $\theta \in [0, 1]$: 

$$f((1 - \theta)x + \theta y) \leq (1 - \theta)f(x) + \theta f(y) - \frac{m}{2} \theta(1 - \theta) \|x - y\|_2^2$$

(a) Prove that a continuously differentiable function $f$ is $m$-strongly convex if and only if for every $x, y \in \mathbb{R}^n$, 

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{m}{2} \|y - x\|_2^2.$$ 

This generalizes the first order characterization of convexity (Section 3.1.3).

(b) Prove that a twice continuously differentiable function $f$ is $m$-strongly convex if and only if its domain and convex and for every $x \in \mathbb{R}^n$, all eigenvalues of the Hessian at $x$ are greater or equal to $m$, i.e.:

$$\nabla^2 f(x) \succeq mI.$$ 

This generalizes the second order characterization of convexity (Section 3.1.4) and is the definition used in Section 9.1.2.

(c) Provide an example of a function that is strongly convex but not everywhere differentiable.

(d) Let $f$ be a $m$-strongly convex function, and $x^*$ an optimum for $\min_{x \in \mathbb{R}^n} f(x)$. Prove that for any point $x \in \mathbb{R}^n$:

$$f(x) \geq f(x^*) + \frac{m}{2} \|x - x^*\|_2^2.$$ 

Conclude that the optimum is unique and that any $\epsilon$-suboptimal point must be close to the optimum. Provide an explicit upper bound on $\|x - x^*\|_2$ for an $\epsilon$-suboptimal $x$. (Note that if $f$ is convex but not strongly convex, $\epsilon$-suboptimal points can be arbitrarily far away from the closest optimum).

Recommended review exercises from Boyd and Vandenberghe (please do not turn these in—they will not be graded): 2.12, 2.15, 3.6, 3.16, 3.18, 3.24, 3.26.