CMSC 39600 - Lec #6 (Oct 11)

Today
- Dinur's proof
- Expanders- properties

Recall last lecture.

PCP Theorem: $\exists$ constant $0<\alpha<1$, & a finite alphabet Σ, s.t.

$$3\text{COLOR}.\cancel{NP} \subseteq PCP^{\Sigma}_{1,1-\alpha}[O(\log n), 2]$$

~~Follows from NP- definition~~

Alternate view of $PCP^{\Sigma}_{1,1-\varepsilon}[r, 2]$

~~3COLOR~~

Constraint graph $(CG)_\beta$

Instance: $\langle G, \Sigma, \mathcal{C} \rangle$

$G = (V, E)$ - graph

Σ - finite alphabet

$\mathcal{C} : \{c_e : \Sigma \times \Sigma \to \{0,1\} \mid e \in E\}$

- constraints on edges

Qn : ~~Is~~ $\exists\ \pi : V \to \Sigma$ s.t. $\forall e,\ c_e(\pi(v_1), \pi(v_2)) = 1$

$e = (v_1, v_2)$

YES: $\exists\ \pi : V \to \Sigma,\ \forall e=(u,v)\ c_e(\pi(v), \pi(u)) = 1$

NO: $\forall\ \pi : V \to \Sigma,\ \dfrac{\#\{e \mid c_e(\pi(v), \pi(u)) = 1\}}{|E|} \leq \beta$

size$(G) = \Theta(m)$

UNSAT(G)

Trivial:

~~3AP~~ CG_{1-1/n^2} is NP-hard

or

3-COLOR $\in PCP^{\Sigma}_{1,1-\frac{1}{n^2}}[2\log n, 2]$, $\forall \Sigma$, $|\Sigma| \geq 3$.

[Gap Amplification Lemma]

$\exists$ Σ and $0 < \alpha < 1$ s.t

$$PCP^{\Sigma}_{1,1-\varepsilon}[r,q] \subseteq PCP^{\Sigma}_{1,1-\varepsilon'}[r+O(1), q]$$

where $\varepsilon' = \min(2\varepsilon, \alpha)$

OR.

There exists a poly time redn from $CG_{1-\varepsilon}$ to $CG_{1-\varepsilon'}$ s.t

$$\langle G, \Sigma, \mathcal{C}_E \rangle \longmapsto \langle G', \Sigma, \mathcal{C} \rangle$$

size $(G') = O(\text{size}(G))$

$UNSAT(G) = 0 \Rightarrow UNSAT(G') = 0$

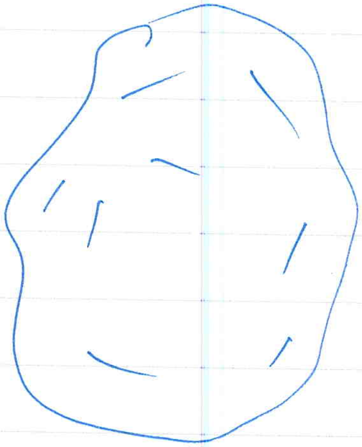
$UNSAT(G') \geq \min(2UNSAT(G'), \alpha)$

Applying Gap Amplification Lemma $O(\log n)$ times

$$PCP^{\Sigma}_{1,1-1/n^2}[O(\log n), 2] \subseteq PCP^{\Sigma}_{1,1-\alpha}[O(\log n), 2]$$

- proving the PCP Theorem.

$UNSAT(G) = \varepsilon$



Fix some $\pi : V \longrightarrow \Sigma$

π violates a random edge w/p ε.

$$\Pr_{e=(u,v)} \left[c_e(\pi(u), \pi(v)) \neq 1 \right] \geq \varepsilon .$$

<u>Sequential Repetition</u>

$$\Pr_{\substack{e_1 \ldots e_t \\ (u_i, v_i)}} \left[\exists i ;\ c_{e_i}(\pi(u_i), \pi(v_i)) = 0 \right] \geq 1 - (1-\varepsilon)^t$$

* GOOD :- UNSAT $\uparrow$
* BAD : Randomness expensive.

If G ~ w. Randomness Efficient ?

~~choose~~

- Suppose G were an expander
- $e_1 \ldots e_t$ - can be chosen by a random walk of length t on expander
- # randomness = $O(\log n) + t \log \deg(G)$

[Req. G - constant deg expander]

Alphabet size - Σ^{d^t}

- can handle this

Three stages

Preprocessing

$G \longrightarrow G_1'$

G' constant degree expander

$\text{size}(G') = O(\text{size}(G))$

$$\text{UNSAT}(G') \geq \frac{\text{UNSAT}(G)}{\beta}$$

Powering

$G_1' \longrightarrow G_2'$

G_2' — t-step walk

$\text{size}(G_2') = O(\text{size}(G_1'))$

$\text{UNSAT}(G') \geq \sqrt{t} \cdot \text{UNSAT}(G)$

$\Sigma' \longrightarrow \Sigma^{dt}$

Composition (alphabet reduction)

Expanders:

Edge Expansion:

$$\varphi(E) = \min_{|S| \leq |V|/2} \frac{E(S, \bar{S})}{|S|}.$$

Spectral expansion

~~(n,d,λ)~~ - G - d-regular

$$\lambda = \max \{\lambda_2, |\lambda_n|\}$$

(n, d, λ)-expander

Lemma: $\cancel{\frac{\varphi^2(G)}{2} \le d-\lambda \le 2\varphi(G)}$

Pf:

Rayleigh Coefficient

$$\lambda = \max_{\substack{x \in \mathbb{R}^n;\ \cancel{x\cdot 1}=0 \\ x \neq 0}} \frac{|\langle Ax, x\rangle|}{\langle x, x\rangle}$$

$$x = \sum a_i v_i$$

$$Ax = \sum a_i \lambda_i v_i.$$

$$\langle Ax, x\rangle = \sum a_i^2 \lambda_i \le \lambda^2 \sum a_i^2$$

Lemma G is d-regular & H is d'-regular

$$G' = G \cup H$$

$$\lambda(G') \le \lambda(G) + \lambda(H)$$

Lemma: $\frac{\varphi^2(G)}{2} \le d-\lambda \le 2\varphi(G)$

Pf: $$x_v = \begin{cases} -|\bar{S}| & v \in S \\ |S| & v \in \bar{S} \end{cases}$$

$$\langle Ax, x\rangle \le \lambda \|x\|^2$$

$$\langle Ax, x\rangle = 2\sum_{(u,v)\in E} x_u x_v = d\,|S||\bar{S}|n - E(S,\bar{S})/n^2$$

$$\|x\|^2 = |S||\bar{S}|n$$

Rapid Margulis / Gaber-Galil Expander

$V = \mathbb{Z}_n \times \mathbb{Z}_n$

$$(x,y) \longrightarrow \begin{cases} (x+2y,\, y) & (x,\, 2x+y) \\ (x+2y+1,\, y) & (x,\, 2x+y+1) \end{cases}$$

8-regular

$\lambda(G) \leq 5\sqrt{2} < 8$

LPS graph:

$V = \mathbb{Z}_p \cup \{\infty\}$

$\infty \cdot 0 = 1$

$\infty + x = \infty$

x — $x+1$

x — $x-1$

x — x^{-1}

$\lambda(G) \leq \lambda_0 < 3$