

---

# Multi-Label Prediction via Compressed Sensing

---

**Daniel Hsu**

University of California, San Diego

DJHSU@CS.UCSB.EDU

**Sham M. Kakade**

Toyota Technological Institute at Chicago

SHAM@TTI-C.ORG

**John Langford**

Yahoo! Research

JL@HUNCH.NET

**Tong Zhang**

Rutgers University

TONGZ@RCI.RUTGERS.EDU

## Abstract

We consider multi-label prediction problems with large output spaces under the assumption of *output sparsity* – that the target vectors have small support. We develop a general theory for a variant of the popular ECOC (error correcting output code) scheme, based on ideas from compressed sensing for exploiting this sparsity. The method can be regarded as a simple reduction from multi-label regression problems to binary regression problems. It is shown that the number of subproblems need only be logarithmic in the total number of label values, making this approach radically more efficient than others. We also state and prove performance guarantees for this method, and test it empirically.

## 1. Introduction

Suppose we have a large database of images, and we want to learn to predict which objects are in any given one. A standard approach to this task is to collect a sample of these images  $x$  along with corresponding labels  $y = (y_1, \dots, y_d) \in \{0, 1\}^d$ , where  $y_i = 1$  if and only if object  $i$  is pictured in  $x$ , and then feed the labeled sample to a multi-label learning algorithm. Here,  $d$  is the total number of objects depicted in the entire database. When  $d$  is very large (e.g.  $10^3$ ,  $10^4$ ), the simple one-against-all approach of learning a sin-

gle predictor for each object  $i = 1, \dots, d$  can become prohibitively expensive at both training and test time.

Our motivation for the present work comes from the observation that although the output (label) space may be very high dimensional, the actual labels are often sparse. That is, in each image, only a small number of objects may be present and there may only be a small amount of ambiguity in what they are. In this work, we consider how this sparsity in the output space, or *output sparsity*, eases the burden of multi-label learning.

**Exploiting output sparsity.** A subtle but critical point that distinguishes output sparsity from more common notions of sparsity (say, in feature or weight vectors) is that we are interested in sparsity in  $\mathbb{E}[y|x]$  rather than  $y$ . In general,  $\mathbb{E}[y|x]$  may be sparse while the actual outcome  $y$  may not (e.g. if there is much unbiased noise); and, vice versa,  $y$  may be sparse with probability one but  $\mathbb{E}[y|x]$  may have full support.

Conventional linear algebra suggests that we must predict  $d$  parameters in order to find the value of  $\mathbb{E}[y|x]$  for each  $x$ . A crucial observation – central to the area of compressed sensing (Donoho, 2006) – is that methods exist to recover  $\mathbb{E}[y|x]$  from just  $O(k \log d)$  measurements when  $\mathbb{E}[y|x]$  is  $k$ -sparse. This is the basis of our approach.

**Our contributions.** We show how to apply algorithms for compressed sensing to the output coding approach (Dietterich & Bakiri, 1995). At a high level, the output coding approach creates a collection of subproblems of the form “Is the label in this subset or its complement?”, solves these problems, and then uses their solution to predict the final label.

---

Preliminary work. Under review by the International Conference on Machine Learning (ICML). Do not distribute.

Our application of compressed sensing is distinct from its more conventional uses for data dimension reduction. Although we do employ a sensing matrix to compress training data, we ultimately are not interested in recovering data explicitly compressed this way. Rather, we learn to predict compressed label vectors, then use sparse reconstruction algorithms to recover uncompressed labels from these predictions. Thus we are interested in reconstruction accuracy of predictions, averaged over the data distribution.

The main contributions of this work are:

1. A formal application of compressed sensing to prediction problems with output sparsity.
2. An efficient output coding method, in which the number of required predictions is only logarithmic in the number of labels  $d$ , making it applicable to very large-scale problems.
3. Robustness guarantees, in the form of regret transform bounds (in general) and a further detailed analysis for the linear prediction setting.

**Prior work.** The idea of output coding was introduced in (Dietterich & Bakiri, 1995) and shown to be useful experimentally. Relative to this work, we expand the scope of the approach to multi-label prediction and provide bounds on regret and error which guide the design of codes.

The loss based decoding approach (Allwein et al., 2000) suggests decoding so as to minimize loss. However, it does not provide significant guidance in the choice of encoding technique, or the feedback between encoding and decoding which we analyze here.

The output coding approach is inconsistent when classifiers are used and the underlying problems being encoded are noisy. This is proved and analyzed in (Langford & Beygelzimer, 2005), where it is shown that using a Hadamard code creates a robust consistent predictor when reduced to binary regression. The approaches here require exponentially (in  $d$ ) fewer predictors. The robustness guarantees here are weaker by a constant factor for the single label case, but stronger for the multi-label case. The multi-label case wasn't specifically analyzed for PECOC, instead the arbitrary cost-sensitive case was analyzed. Carefully tuning the SECOC algorithm there gives a  $k^2$  dependence in the regret bound, which is functionally worse than the bounds here.

Our algorithms rely on several compressed sensing approaches, which we detail where used.

## 2. Preliminaries

Let  $\mathcal{X}$  be an arbitrary input space and  $\mathcal{Y} \subset \mathbb{R}^d$  be a  $d$ -dimensional output (label) space. We assume the data source is defined by a fixed but unknown distribution over  $\mathcal{X} \times \mathcal{Y}$ . Our goal is to learn a predictor  $F : \mathcal{X} \rightarrow \mathcal{Y}$  with low expected  $\ell_2^2$ -error

$$\mathbb{E}_x \|F(x) - \mathbb{E}[y|x]\|_2^2$$

(the sum of mean-squared-errors over all labels) using a random i.i.d. sample of  $n$  data  $\{(x_i, y_i)\}_{i=1}^n$ .

We focus on the regime in which the output space is very high-dimensional ( $d$  very large), but for any given  $x \in \mathcal{X}$ , the expected value  $\mathbb{E}[y|x]$  of the corresponding label  $y \in \mathcal{Y}$  has only a few non-zero entries. A vector is  $k$ -sparse if it has at most  $k$  non-zero entries.

## 3. Learning and Prediction

### 3.1. Learning Compressed Labels

Let  $A : \mathbb{R}^d \rightarrow \mathbb{R}^m$  be a linear compression function, where  $m \leq d$  (but hopefully  $m \ll d$ ). We use  $A$  to compress (i.e. reduce the dimension of) the labels  $\mathcal{Y}$ , and learn a predictor  $H : \mathcal{X} \rightarrow A(\mathcal{Y})$  of these compressed labels. Since  $A$  is linear, we simply represent  $A \in \mathbb{R}^{m \times d}$  as a matrix.

Specifically, given a sample  $\{(x_i, y_i)\}_{i=1}^n$ , we form a compressed sample  $\{(x_i, Ay_i)\}_{i=1}^n$  and then learn a predictor  $H$  of  $\mathbb{E}[Ay|x] = A\mathbb{E}[y|x]$  with the objective of minimizing the  $\ell_2^2$ -error

$$\mathbb{E}_x \|H(x) - \mathbb{E}[Ay|x]\|_2^2.$$

### 3.2. Prediction

To obtain a predictor  $F$  of  $\mathbb{E}[y|x]$ , we compose the predictor  $H$  of  $\mathbb{E}[Ay|x]$  (learned using the compressed sample) with a reconstruction algorithm  $R : \mathbb{R}^m \rightarrow \mathbb{R}^d$ . The algorithm  $R$  maps predictions of compressed labels  $Ay \in A\mathcal{Y}$  to predictions of labels  $y \in \mathcal{Y}$  in the original output space.

Recent developments in the area of compressed sensing have produced a spate of algorithms  $R$  with strong performance guarantees when the compression function  $A$  satisfies certain properties. We abstract out the relevant aspects of these guarantees in the following definition.

**Definition.** An algorithm  $R$  is a *valid reconstruction algorithm for a family of compression functions*  $\{\mathcal{A}_k \subset \bigcup_{m \geq 1} \mathbb{R}^{m \times d} : k \in \mathbb{N}\}$  and *sparsity error*  $\text{sperr} : \mathbb{N} \times \mathbb{R}^d \rightarrow \mathbb{R}$ , if there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  and constants  $C_1, C_2 \in \mathbb{R}$  such that: on input  $k \in \mathbb{N}, A \in$

---

**Algorithm 1** Training algorithm

**parameters** sparsity level  $k$ , regression learning algorithm  $L$ , compression function  $A \in \mathbb{R}^{m \times d}$   
**input** training data  $S \subset \mathcal{X} \times \mathbb{R}^d$   
**for**  $i = 1, \dots, m$  **do**  
      $h_i \leftarrow L(\{(x, (Ay)_i) : (x, y) \in S\})$   
**end for**  
**output** regressors  $H = [h_1, \dots, h_m]$

---

$\mathcal{A}_k$  with  $m$  rows, and  $h \in \mathbb{R}^m$ , the algorithm  $R(k, A, h)$  returns an  $f(k)$ -sparse vector  $\hat{y}$  satisfying

$$\|\hat{y} - y\|_2^2 \leq C_1 \cdot \|h - Ay\|_2^2 + C_2 \cdot \text{sperr}(k, y)$$

for all  $y \in \mathbb{R}^d$ . The function  $f$  is the *output sparsity* of  $R$  and the constants  $C_1$  and  $C_2$  are the *regret factors*.

Loosely put, a reconstruction algorithm should be agnostic about the sparsity of the original signal ( $\text{sperr}(k, \mathbb{E}[y|x])$ ), as well as the “measurement noise” (the prediction error  $\|H(x) - \mathbb{E}[Ay|x]\|_2$ ).

We make a few additional remarks on the definition.

1. The minimum number of rows of matrices  $A \in \mathcal{A}_k$  may in general depend on  $k$  (as well as the ambient dimension  $d$ ). In the next section, we show how to construct such  $A$  with close to the optimal number of rows.
2. The sparsity error  $\text{sperr}(k, y)$  should measure how poorly  $y \in \mathbb{R}^d$  is approximated by a  $k$ -sparse vector.
3. A reasonable output sparsity  $f(k)$  for sparsity level  $k$  should not be much more than  $k$ , e.g.  $f(k) = O(k)$ .

Concrete examples of valid reconstruction algorithms (along with the associated  $\mathcal{A}_k$ ,  $\text{sperr}$ , etc.) are given in the next section.

## 4. Algorithms

Our prescribed recipe is summarized in Algorithms 1 and 2. We give some examples of compression functions and reconstruction algorithms in the following subsections.

### 4.1. Compression Functions

Several valid reconstruction algorithms are known for compression matrices that satisfy a *restricted isometry property*.

---

**Algorithm 2** Prediction algorithm

**parameters** sparsity level  $k$ , compression function  $A \in \mathcal{A}_k$ , valid reconstruction algorithm  $R$  for  $\mathcal{A}_k$ ,  
**input** regressors  $H = [h_1, \dots, h_m]$ , test point  $x \in \mathcal{X}$   
**output**  $\hat{y} = \vec{R}(k, A, [h_1(x), \dots, h_m(x)])$

---



---

**Algorithm 3** Prediction algorithm with  $R = \text{OMP}$ 

**parameters** sparsity level  $k$ , compression function  $A = [a_1 | \dots | a_d] \in \mathcal{A}_k$ ,  
**input** regressors  $H = [h_1, \dots, h_m]$ , test point  $x \in \mathcal{X}$   
 $b \leftarrow [h_1(x), \dots, h_m(x)]^\top$   
 $\hat{y} \leftarrow \vec{0}$ ,  $J \leftarrow \emptyset$ ,  $r \leftarrow b$   
**for**  $i = 1, \dots, 2k$  **do**  
      $j_* \leftarrow \arg \max_j |r^\top a_j|$   
      $J \leftarrow J \cup \{j_*\}$   
      $\hat{y}_J \leftarrow (A_J)^\dagger b$  (least-squares restricted to  $J$ )  
      $\hat{y}_{J^c} \leftarrow \vec{0}$   
      $r \leftarrow b - A\hat{y}$   
**end for**  
**output**  $\hat{y}$

---

**Definition.** A matrix  $A \in \mathbb{R}^{m \times d}$  satisfies the  $(k, \delta)$ -restricted isometry property ( $(k, \delta)$ -RIP),  $\delta \in (0, 1)$ , if

$$(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2$$

for all  $k$ -sparse  $x \in \mathbb{R}^d$ .

While some explicit constructions of  $(k, \delta)$ -RIP matrices are known (e.g. ( DeVore, 2007)), the best guarantees are obtained when the matrix is chosen randomly from an appropriate distribution, such as one of the following (Mendelson et al., 2008; Rudelson & Vershynin, 2006).

- All entries i.i.d. Gaussian  $N(0, 1/m)$ , with  $m = O(k \log(d/k))$ .
- All entries i.i.d. Bernoulli  $B(1/2, 1/2)$  over  $\{\pm 1/\sqrt{m}\}$ , with  $m = O(k \log(d/k))$ .
- Random subset of rows from the  $d \times d$  Hadamard matrix over  $\{\pm 1/\sqrt{m}\}$ , with  $m = O(k \log^5 d)$ .

The hidden constants in the big- $O$  notation depend inversely on  $\delta$  and the probability of success.

A striking feature of these constructions is the very mild dependence of  $m$  on the ambient dimension  $d$ . This translates to a significant savings in the number of learning problems one has to solve after employing our reduction.

Some reconstruction algorithms require a stronger guarantee of bounded *coherence*  $\mu(A) \leq O(1/k)$ ,

where  $\mu(A)$  defined as

$$\mu(A) = \max_{1 \leq i < j \leq d} |(A^\top A)_{i,j}| / \sqrt{|(A^\top A)_{i,i}| |(A^\top A)_{j,j}|}$$

It is easy to check that the Gaussian, Bernoulli, and Hadamard-based random matrices given above have coherence bounded by  $O(\sqrt{(\log d)/m})$  with high probability. Thus, one can take  $m = O(k^2 \log d)$  to guarantee  $1/k$  coherence. This is a factor  $k$  worse than what was needed for  $(k, \delta)$ -RIP, but the dependence on  $d$  is still small.

## 4.2. Reconstruction Algorithms

In Table 1, we give some examples of valid reconstruction algorithms. These algorithms are valid for the  $\mathcal{A}_k$  listed in the table, and sparsity error

$$\text{sperr}(k, y) = \|y - y_{(1:k)}\|_2^2 + \frac{1}{k} \|y - y_{(1:k)}\|_1^2$$

where  $y_{(1:k)}$  is the best  $k$ -sparse approximation of  $y$  (i.e. the vector with just the  $k$  largest (in magnitude) coefficients of  $y$ ).

The following theorem relates reconstruction quality to approximate sparse regression, giving a sufficient condition for an algorithm to be valid for RIP matrices.

**Theorem 1.** *Let  $\mathcal{A}_k = \{(k + f(k), \delta)\text{-RIP matrices}\}$  for some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , and let  $A \in \mathcal{A}_k$  have  $m$  rows. If for any  $h \in \mathbb{R}^m$ , a reconstruction algorithm  $R$  returns an  $f(k)$ -sparse solution  $\hat{y} = R(k, A, h)$  satisfying*

$$\|A\hat{y} - h\|_2^2 \leq \inf_{y \in \mathbb{R}^d} C \|Ay_{(1:k)} - h\|_2^2,$$

then it is a valid reconstruction algorithm for  $\mathcal{A}_k$  and  $\text{sperr}$  given above, with output sparsity  $f$  and regret factors  $C_1 = 2(1 - \delta)^{-1}(1 + \sqrt{C})^2$  and  $C_2 = 4(1 - \delta)^{-2}(2 + \sqrt{C})^2$ .

We defer all proofs to the appendices.

**Iterative and greedy algorithms.** Orthogonal Matching Pursuit (OMP) (Mallat & Zhang, 1993), FoBa (Zhang, 2008), and CoSaMP (Needell & Tropp, 2007) are examples of iterative or greedy reconstruction algorithms. OMP is a greedy forward selection method that repeatedly selects a new column of  $A$  to use in fitting  $h$  (see Algorithm 3). FoBa is similar, except it also incorporates backward steps to un-select columns that are later discovered to be unnecessary. CoSaMP is also similar to OMP, but instead selects larger sets of columns in each iteration.

The validity of FoBa and CoSaMP are apparent from the cited references. For OMP, we give the following guarantee.

Table 1. Examples of valid reconstruction algorithms.

Algorithm	$\mathcal{A}_k$	$f(k)$
BP/Lasso	(see text)	-
OMP	$\mu(A) \leq 0.1/k$	$2k$
FoBa	$(8k, 0.1)$ -RIP	$8k$
CoSaMP	$(4k, 0.1)$ -RIP	$2k$

**Theorem 2.** *If  $\mu(A) \leq 0.1/k$ , then after  $f(k) = 2k$  steps of OMP, the algorithm returns  $\hat{y}$  satisfying*

$$\|A\hat{y} - h\|_2^2 \leq 23 \|Ay_{(1:k)} - h\|_2^2 \quad (\forall y \in \mathbb{R}^d).$$

$\ell_1$  **algorithms.** Basis Pursuit (BP) (Candès et al., 2006) and its variants are based on finding the minimum  $\ell_1$ -norm solution to a linear system. While the basic form of BP is ill-suited for our application (it requires the user to supply the amount of measurement error  $\|Ay - h\|_2$ ), its more advanced path-following or multi-stage variants may be valid. We use the popular LARS/Lasso algorithm (Efron et al., 2004) in our experiments.

## 5. Analysis

### 5.1. Regret Analysis

We now state our main regret transform bound, which follows immediately from the definition of a valid reconstruction algorithm and linearity of expectation.

**Theorem 3 (Regret Transform).** *Let  $R$  be a valid reconstruction algorithm for  $\{\mathcal{A}_k : k \in \mathbb{N}\}$  and  $\text{sperr} : \mathbb{N} \times \mathbb{R}^d \rightarrow \mathbb{R}$ . Then there exists some constants  $C_1$  and  $C_2$  such that the following holds. Pick any  $k \in \mathbb{N}$ ,  $A \in \mathcal{A}_k$  with  $m$  rows, and  $H : \mathcal{X} \rightarrow \mathbb{R}^m$ . Let  $F : \mathcal{X} \rightarrow \mathbb{R}^d$  be the composition of  $R(k, A, \cdot)$  and  $H$ , i.e.  $F(x) = R(k, A, H(x))$ . Then*

$$\mathbb{E}_x \|F(x) - \mathbb{E}[y|x]\|_2^2 \leq C_1 \cdot \mathbb{E}_x \|H(x) - \mathbb{E}[Ay|x]\|_2^2 + C_2 \cdot \text{sperr}(k, \mathbb{E}[y|x]).$$

In order compare regret bound in Theorem 3 with the bounds afforded by Sensitive Error Correcting Output Codes (SECO) (Langford & Beygelzimer, 2005), we need to relate  $\mathbb{E}_x \|H(x) - \mathbb{E}[Ay|x]\|_2^2$  to the average scaled mean-squared-error over all induced regression problems, where error is scaled by the maximum difference  $L_i = \max_y (Ay)_i - \min_y (Ay)_i$  between induced labels:

$$\bar{r} = \frac{1}{m} \sum_{i=1}^m \mathbb{E}_x \left( \frac{H(x)_i - \mathbb{E}[(Ay)_i|x]}{L_i} \right)^2.$$

In these terms, SECOC can be tuned to yield

$$\mathbb{E}_x \|F(x) - \mathbb{E}[y|x]\|_2^2 \leq 4k^2 \cdot \bar{r}$$

for general  $k$ .

For now, ignore the sparsity error. Let  $A \in \mathbb{R}^{m \times d}$  with  $N(0, 1/m)$  entries, where  $m = O(k \log d)$ . One can show that with high probability over the choice of  $A$ ,  $\sup_y \|Ay\|_\infty = O(\sqrt{k \log(m)/m})$ , where the supremum is taken over  $k$ -sparse vectors  $y \in \{0, 1\}^d$  (see Lemma 8 in Appendix C). In this event, we have  $L_i = O(\sqrt{k \log(m)/m})$ , so we have the bound

$$\begin{aligned} C_1 \cdot \mathbb{E}_x \|H(x) - \mathbb{E}[Ay|x]\|_2^2 \\ \leq O(k \log m) \cdot \bar{r} = O(k \log k + k \log \log d) \cdot \bar{r}. \end{aligned}$$

The improved dependence on  $k$  is at the cost of larger constants and  $\log \log d$ .

Now we consider the sparsity error. When  $m = d$  (as is the case for SECOC),  $\mathbb{E}[y|x]$  is allowed to be fully dense ( $k = d$ ), in which case  $\text{sperr}(k, \mathbb{E}[y|x]) = 0$ . Thus we pay a penalty for inducing just  $k < d$  problems unless  $\mathbb{E}[y|x]$  is truly  $k$ -sparse.

## 5.2. Linear Prediction

A danger of using generic reductions is that one creates a problem instance that is even harder to solve than the original problem. This is an oft cited issue with using output codes for multi-class problems.

In the case of linear prediction, however, the danger is mitigated. As a warm up, suppose there is a perfect linear predictor of  $\mathbb{E}[y|x]$ , i.e.  $\mathbb{E}[y|x] = B^\top x$  for some  $B \in \mathbb{R}^{p \times d}$  (here  $\mathcal{X} = \mathbb{R}^p$ ). Then it is easy to see that  $H = BA^\top$  is a perfect linear predictor of  $\mathbb{E}[Ay|x]$ :  $H^\top x = AB^\top x = A\mathbb{E}[y|x] = \mathbb{E}[Ay|x]$ . The following theorem generalizes this observation to imperfect linear predictors for certain well-behaved  $A$ .

**Theorem 4.** *Suppose  $\mathcal{X} \subset \mathbb{R}^p$ . Let  $B \in \mathbb{R}^{p \times d}$  be a linear function with*

$$\mathbb{E}_x \|B^\top x - \mathbb{E}[y|x]\|_2^2 = \epsilon.$$

*Let  $A \in \mathbb{R}^{m \times d}$  have entries drawn i.i.d. from  $N(0, 1/m)$ , and let  $H = BA^\top$ . Then with high probability (over the choice of  $A$ ),*

$$\mathbb{E}_x \|H^\top x - A\mathbb{E}[y|x]\|_2^2 \leq (1 + O(1/\sqrt{m})) \epsilon.$$

**Remark 5.** *Similar guarantees can be proven for the Bernoulli-based matrices. Note that  $d$  does not appear in the bound. This is in contrast to the expected spectral norm of  $A$ , which is  $1 + O(\sqrt{d/m})$ .*

Theorem 4 implies that the errors of *any* linear predictor are not magnified much by the compression function. So a good linear predictor for the original problem implies an almost-as-good linear predictor for the induced problem.

## 6. Experiments

We conducted an empirical assessment of the proposed reduction on two labeled data sets with large label spaces. These experiments demonstrate the feasibility of the method – a sanity check that the reduction does in fact preserve learnability – and compare different compression and reconstruction options. However, the scale of these experiments, in terms of data size and predictor complexity, is not yet able to show computational savings in terms of wall-clock measurements.

### 6.1. Data

**Image data.**<sup>1</sup> The first data set was collected by the ESP Game (von Ahn & Dabbish, 2004), an online game in which players ultimately provide word tags for a diverse set of web images.

The set contains nearly 68000 images, with about 22000 unique labels. We retained just the 1000 most frequent labels: the least frequent of these occurs 39 times in the data, and the most frequent occurs about 12000 times. Each image contains about four labels on average. We used half of the data for training and half for testing.

We represented each image as a bag-of-features vector in a manner similar to (Marszalek et al., 2007). Specifically, we identified 1024 representative SURF features points (Bay et al., 2008) from  $10 \times 10$  gray-scale patches chosen randomly from the training images; this partitions the space of image patches (represented with SURF features) into Voronoi cells. We then built a histogram for each image, counting the number of patches that fall in each cell.

**Text data.**<sup>2</sup> The second data set was collected by Tsoumakas et al. (2008) from `del.icio.us`, a social bookmarking service in which users assign descriptive textual tags to web pages.

The set contains about 16000 labeled web page and 983 unique labels. The least frequent label occurs 21 times and the most frequent occurs almost 6500 times. Each web page is assigned 19 labels on average. Again, we used half the data for training and half for testing.

<sup>1</sup><http://hunch.net/~learning/ESP-ImageSet.tar.gz>

<sup>2</sup><http://mlkd.csd.auth.gr/multilabel.html>

Each web page is represented as a boolean bag-of-words vector, with the vocabulary chosen using a combination of frequency thresholding and  $\chi^2$  feature ranking. See (Tsoumakas et al., 2008) for details.

Each binary label vector (for both data sets) has ones in the coordinates corresponding to the labels assigned to the data point.

## 6.2. Output Sparsity

We first performed a bit of exploratory data analysis to get a sense of how sparse the target in our data is. We computed the least-squares linear regressor  $\hat{B} \in \mathbb{R}^{p \times d}$  on the training data (without any output coding) and predicted the label probabilities  $\hat{p}(x) = \hat{B}^\top x$  on the test data (clipping values to the range  $[0, 1]$ ). Using  $\hat{p}(x)$  as a surrogate for the actual target  $\mathbb{E}[y|x]$ , we examined the relative  $\ell_2^2$  error of  $\hat{p}$  and its best  $k$ -sparse approximation  $\epsilon(k, \hat{p}(x)) = \sum_{i=k+1}^d \hat{p}_{(i)}(x)^2 / \|\hat{p}(x)\|_2^2$ , where  $\hat{p}_{(1)}(x) \geq \dots \geq \hat{p}_{(d)}(x)$ .

Examining  $\mathbb{E}_x \epsilon(k, \hat{p}(x))$  as a function of  $k$ , we saw that in both the image and text data, the fall-off with  $k$  is eventually super-polynomial, but we are interested in the behavior for small  $k$  where it appears polynomial  $k^{-r}$  for some  $r$ . Around  $k = 10$ , we estimated an exponent of 0.50 for the image data and 0.55 for the text data. This is somewhat below the standard of what is considered sparse (e.g. vectors with small  $\ell_1$ -norm show  $k^{-1}$  decay). Thus, we expect the reconstruction algorithms will have to contend with the sparsity error of the target.

## 6.3. Procedure

We used least-squares linear regression as our base learning algorithm, with no regularization on the image data and with ridge regularization with the text data ( $\lambda = 0.01$ ) for numerical stability reasons. We did not attempt any parameter tuning.

The compression functions we used were generated by selecting  $m$  random rows of the  $1024 \times 1024$  Hadamard matrix, for  $m \in \{100, 200, 300, 400\}$ . We also experimented with Gaussian matrices, but these uniformly yielded worse results.

We tested the greedy reconstruction algorithms described earlier (OMP, FoBa, and CoSaMP) as well as a path-following version of Lasso based on LARS (Efron et al., 2004). Each algorithm was used to recover a  $k$ -sparse label vector  $\hat{y}^k$  from the predicted compressed label  $H(x)$ , for  $k = 1, \dots, 10$ . We measured the  $\ell_2^2$  distance  $\|\hat{y}^k - y\|_2^2$  of the prediction to the true test label  $y$ . In addition, we measured the precision of the predicted

support at various values of  $k$  using the 10-sparse label prediction. That is, we ordered the coefficients of each 10-sparse label prediction  $\hat{y}^{10}$  by magnitude, and measured the precision of predicting the first  $k$  coordinates  $|\text{supp}(\hat{y}_{(1:k)}^{10}) \cap \text{supp}(y)|/k$ . Actually, for  $k \geq 6$ , we used  $\hat{y}^{2k}$  instead of  $\hat{y}^{10}$ .

We used correlation decoding (CD) as a baseline method, as it is a standard decoding method for ECOC approaches. CD predicts using the top  $k$  coordinates in  $A^\top H(x)$ , ordered by magnitude. For mean-squared-error comparisons, we used the least-squares approximation of  $H(x)$  using these  $k$  columns of  $A$ . In contrast to the other algorithms, CD does not enjoy the robustness guarantees required by our reduction.

## 6.4. Results

As expected, the performance of the reduction, using any reconstruction algorithm, improves as the number of induced subproblems  $m$  is increased (see Figures 1 and 2; at  $m = 300, 400$ , the precision-at- $k$  is nearly the same as one-against-all, i.e.  $m = 1024$ ). When  $m$  is small and  $A \notin \mathcal{A}_K$ , the reconstruction algorithm cannot reliably choose  $k \geq K$  coordinates, so its performance may degrade after this point by over-fitting. But when the compression function  $A$  is in  $\mathcal{A}_K$  for a sufficiently large  $K$ , then the squared-error decreases as the output sparsity  $k$  increases up to  $K$ . Note the fact that precision-at- $k$  decreases as  $k$  increases is expected, as fewer data will have at least  $k$  correct labels.

All of the reconstruction algorithms at least match or out-performed the baseline on the mean-squared-error criterion, except when  $m = 100$ . When  $A$  has few rows, (1)  $A \in \mathcal{A}_K$  only for very small  $K$ , and (2) many of its columns will have significant correlation. In this case, when choosing  $k > K$  columns, it is better to choose correlated columns to avoid over-fitting. Both OMP and FoBa explicitly avoid this and thus do not fare well; but CoSaMP, Lasso, and CD do allow selecting correlated columns and thus perform better in this regime.

The results for precision-at- $k$  are similar to that of mean-squared-error, except that choosing correlated columns does not necessarily help in the small  $m$  regime. This is because the extra correlated columns need not correspond to accurate label coordinates.

In summary, the experiments demonstrate the feasibility and robustness of our reduction method for two natural multi-label prediction tasks. They show that predictions of relatively few compressed labels are sufficient to recover an accurate sparse label vector, and as our theory suggests, the robustness of the recon-

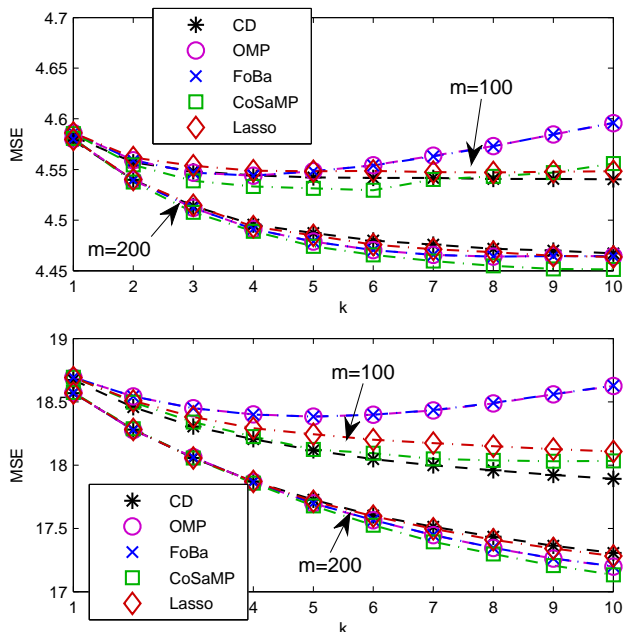


Figure 1. Mean-squared-error versus output sparsity  $k$ ,  $m \in \{100, 200\}$ . Top: image data. Bottom: text data. In each plot: the top set of lines corresponds to  $m = 100$ , and the bottom set to  $m = 200$ .

struction algorithms is a key factor in their success.

## References

- Allwein, E., Schapire, R., & Singer, Y. (2000). Reducing multiclass to binary: A unifying approach for margin classifiers. *Journal of Machine Learning Research*, 1, 113–141.
- Bay, H., Ess, A., Tuytelaars, T., & Gool, L. V. (2008). SURF: Speeded up robust features. *Computer Vision and Image Understanding*, 110, 346–359.
- Candès, E., Romberg, J., & Tao, T. (2006). Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math.*, 59, 1207–1222.
- Dasgupta, S. (2000). *Learning probability distributions*. Doctoral dissertation, University of California.
- DeVore, R. (2007). Deterministic constructions of compressed sensing matrices. *J. of Complexity*, 23, 918–925.
- Dietterich, T., & Bakiri, G. (1995). Solving multiclass learning problems via error-correcting output codes. *Journal of Artificial Intelligence Research*, 2, 263–286.
- Donoho, D. (2006). Compressed sensing. *IEEE Trans. Info. Theory*, 52, 1289–1306.
- Donoho, D., Elad, M., & Temlyakov, V. (2006). Stable recovery of sparse overcomplete representations in the presence of noise. *IEEE Trans. Info. Theory*, 52, 6–18.

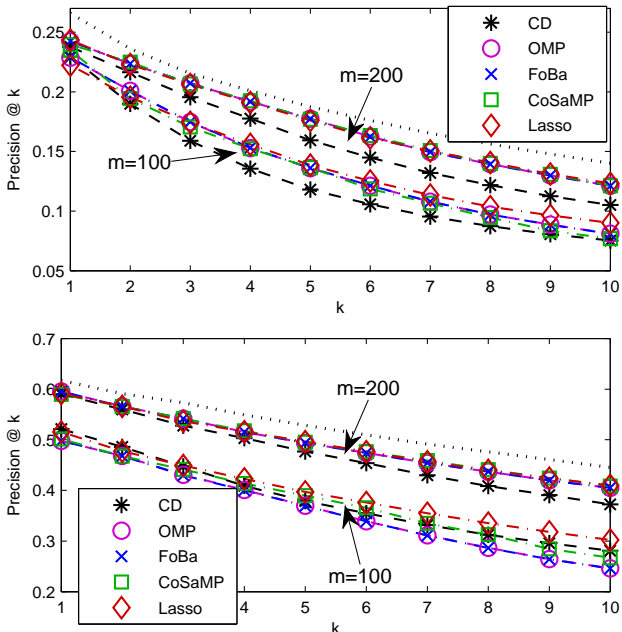


Figure 2. Mean precision-at- $k$  versus output sparsity  $k$ ,  $m \in \{100, 200\}$ . Top: image data. Bottom: text data. In each plot: the top black unadorned line is one-against-all ( $m = 1024$ ), the middle set of lines corresponds to  $m = 200$ , and the bottom set to  $m = 100$ .

- Efron, B., Hastie, T., Johnstone, I., & Tibshirani, R. (2004). Least angle regression. *Annals of Statistics*, 32, 407–499.
- Langford, J., & Beygelzimer, A. (2005). Sensitive error correcting output codes. *Proc. Conference on Learning Theory*.
- Ledoux, M. (2001). *The concentration of measure phenomenon*. American Mathematical Society.
- Mallat, S., & Zhang, Z. (1993). Matching pursuits with time-frequency dictionaries. *IEEE Transactions on Signal Processing*, 41, 3397–3415.
- Marszałek, M., Schmid, C., Harzallah, H., & van de Weijer, J. (2007). Learning object representations for visual object class recognition. *Visual Recognition Challenge Workshop, in conjunction with ICCV*.
- Mendelson, S., Pajor, A., & Tomczak-Jaegermann, N. (2008). Uniform uncertainty principle for Bernoulli and subgaussian ensembles. *Constructive Approximation*, 28, 277–289.
- Needell, D., & Tropp, J. (2007). CoSaMP: Iterative signal recovery from incomplete and inaccurate samples. *Applied and Computational Harmonic Analysis*.
- Rudelson, M., & Vershynin, R. (2006). Sparse reconstruction by convex relaxation: Fourier and Gaussian measurements. *Proc. Conference on Information Sciences and Systems*.

Tsoumakas, G., Katakis, I., & Vlahavas, I. (2008). Effective and efficient multilabel classification in domains with large number of labels. *Proc. ECML/PKDD 2008 Workshop on Mining Multidimensional Data*.

von Ahn, L., & Dabbish, L. (2004). Labeling images with a computer game. *Proc. ACM Conference on Human Factors in Computing Systems*.

Zhang, T. (2008). Adaptive forward-backward greedy algorithm for sparse learning with linear models. *Proc. Neural Information Processing Systems*.

## A. Proof of Theorem 1

Without loss of generality, assume  $|y_1| \geq \dots \geq |y_d|$ . Let  $\ell = k + f(k)$  and  $\Delta = y - y_{(1:k)}$ . We decompose  $\Delta$  as  $\Delta = \sum_{i \geq 0} y_{J_i}$ , where  $J_i = \{k + i\ell + 1, \dots, k + (i + 1)\ell\}$ . Hölder's inequality implies  $\|y_{J_i}\|_2^2 \leq \|y_{J_i}\|_\infty \|y_{J_i}\|_1 \leq \|y_{J_i}\|_\infty \|y_{J_{i-1}}\|_1$ , and for  $i \geq 1$ , we have  $\|y_{J_{i-1}}\|_1 \geq \ell \|y_{J_i}\|_\infty$  by definition. Therefore  $\|y_{J_i}\|_2^2 \leq \|y_{J_{i-1}}\|_1^2 / \ell$  for  $i \geq 1$ . Now exploiting the RIP guarantee for  $A$  and the above inequality, we have

$$\begin{aligned} \|A\Delta\|_2 &\leq \sum_{i \geq 0} \|Ay_{J_i}\|_2 \leq \sqrt{1 + \delta} \sum_{i \geq 0} \|y_{J_i}\|_2 \\ &= \sqrt{1 + \delta} [\|y_{J_0}\|_2 + \sum_{i \geq 1} \|y_{J_i}\|_2] \\ &\leq \sqrt{1 + \delta} [\|y_{J_0}\|_2 + \sum_{i \geq 0} \|y_{J_i}\|_1 / \sqrt{\ell}] \\ &\leq \sqrt{1 + \delta} [\|\Delta\|_2 + \|\Delta\|_1 / \sqrt{\ell}]. \end{aligned}$$

Therefore  $\|Ay_{(1:k)} - h\|_2 \leq \|Ay - h\|_2 + \|A\Delta\|_2 \leq \|Ay - h\|_2 + \sqrt{1 + \delta} [\|\Delta\|_2 + \|\Delta\|_1 / \sqrt{\ell}]$ . Hence

$$\begin{aligned} \|\hat{y} - y\|_2 &\leq \|\hat{y} - y_{(1:k)}\|_2 + \|\Delta\|_2 \\ &\leq (1 - \delta)^{-1/2} \|A(\hat{y} - y_{(1:k)})\|_2 + \|\Delta\|_2 \\ &\leq (1 - \delta)^{-1/2} [\|A\hat{y} - h\|_2 + \|Ay_{(1:k)} - h\|_2] + \|\Delta\|_2 \\ &\leq (1 - \delta)^{-1/2} [(1 + \sqrt{C}) \|Ay_{(1:k)} - h\|_2] + \|\Delta\|_2 \\ &\leq (1 - \delta)^{-1/2} (1 + \sqrt{C}) \|Ay - h\|_2 \\ &\quad + ((1 - \delta)^{-1} (2 + \sqrt{C})) [\|\Delta\|_1 / \sqrt{\ell} + \|\Delta\|_2]. \end{aligned}$$

To conclude, we square both sides and repeatedly simplify with the fact  $(x + y)^2 \leq 2x^2 + 2y^2$ .  $\square$

## B. Proof of Theorem 2

Without loss of generality, we assume that  $[A^T A]_{i,i} = 1$  (this can be achieved by normalizing the columns of  $A$ ) and the support of  $y_{(1:k)}$  is  $\{1 \dots k\}$ . Let  $a_j$  be the  $j$ -th column of  $A$ .

We consider three solution vectors:

1.  $y$ : the  $k$ -sparse solution we want to compare to.
2.  $y'$ : the  $\leq (2k - 1)$ -sparse solution obtained by running OMP starting from  $y$ , with the property that the amount of progress (reduction in squared-error) to be made by another iteration is small: at most  $\|h - Ay\|_2^2 / k$ . Since  $y'$  is obtained by starting with  $y$ , it can only have smaller squared-error.
3.  $\hat{y}^i$ : the actual solution returned by OMP up to the point at which it chooses a column  $j \notin \text{supp}(y')$ . When this happens, we have the guarantee that  $|a_j^\top (h - A\hat{y}^i)| \geq |a_\ell^\top (h - A\hat{y}^i)|$  for all  $\ell \in \text{supp}(y')$ .

Let  $\Delta = \hat{y}^i - y'$ . The goal is to bound  $\|A\Delta\|_2^2$  in terms of  $\|h - Ay'\|_2^2$  and  $|a_j^\top (h - Ay')|^2$ . A key step is in bounding  $|a_j^\top A\Delta|$ , which uses the fact that  $j \notin \text{supp}(\Delta) \subseteq \text{supp}(y')$  and that the coherence of  $A$  is bounded  $\mu(A) \leq 0.1/k$ .

*Proof of Theorem 2.* Consider running  $k$  iterations of OMP starting at the solution  $y^{(0)} = y_{(1:k)}$ , denoting the intermediate solutions by  $y^{(1)}, \dots, y^{(k)}$ . Let  $r^{(i)} = h - Ay^{(i)}$  be the residual after step  $i$ ,  $a_i$  be the additional column of  $A$  included in  $y^{(i)}$ , and  $\rho_i$  be the reduction in squared-error  $\rho_i = \|h - Ay^{(i-1)}\|_2^2 - \|h - Ay^{(i)}\|_2^2 = \|r^{(i-1)}\|_2^2 - \|r^{(i)}\|_2^2$  from the previous step. We want to bound  $\rho_i$  from below. To this end, consider the progress made by using  $\tilde{y}^{(i)} = y^{(i-1)} + \alpha_i a_i$  instead of  $y^{(i)}$ , where  $\alpha_i = (a_i^\top r^{(i-1)})^{-1}$ :

$$\begin{aligned} &\|r^{(i-1)}\|_2^2 - \|h - A\tilde{y}^{(i)}\|_2^2 \\ &= \|r^{(i-1)}\|_2^2 - \|h - (Ay^{(i-1)} + \alpha_i a_i)\|_2^2 \\ &= \|r^{(i-1)}\|_2^2 - \|r^{(i-1)} - \alpha_i a_i\|_2^2 \\ &= \|r^{(i-1)}\|_2^2 - (\|r^{(i-1)}\|_2^2 - 2\alpha_i a_i^\top r^{(i-1)} + \alpha_i^2) \\ &= (\alpha_i^\top r^{(i-1)})^2. \end{aligned}$$

Since  $y^{(i)}$  is the least-squares fit of  $h$  using the same columns of  $A$  as  $\tilde{y}^{(i)}$ , its reduction in error can only be greater. Hence  $\rho_i \geq (\alpha_i^\top r^{(i-1)})^2$ , and thus  $\sum_{i=1}^k \rho_i \geq \sum_{i=1}^k (\alpha_i^\top r^{(i-1)})^2$ . The total reduction in error  $\sum_{i=1}^k \rho_i$  is at most  $\|r^{(0)}\|_2^2$ , so there must be some  $i^*$  for which  $(\alpha_{i^*}^\top r^{(i^*-1)})^2 \leq \|r^{(0)}\|_2^2 / k$ , i.e.

$$(\alpha_{i^*}^\top (h - Ay^{(i^*-1)}))^2 \leq \|h - Ay_{(1:k)}\|_2^2 / k$$

(else we have the contradiction  $\|h - Ay_{(1:k)}\|_2^2 \geq \sum_i \rho_i \geq \sum_i (\alpha_i^\top r^{(i)})^2 > \|h - Ay_{(1:k)}\|_2^2$ ). Let  $y' = y^{(i^*-1)}$  be the solution prior to this step, and let  $a_{i^*}$  be the selected column. It is clear that  $\|h - Ay'\|_2^2 \leq \|h - Ay_{(1:k)}\|_2^2$  (the error could only have improved).

Without loss of generality, let the support of  $y'$  be (a subset of)  $\{1 \dots 2k\}$ .

Now consider the actual execution of OMP (not starting from  $y_{(1:k)}$ ). We will compare the solution returned after  $2k$  steps to the  $y'$  defined above.

Let  $i + 1$  be the first time that OMP selects a column  $j$  not in  $\{1 \dots 2k\}$  (if this never happens, then  $\|h - A\hat{y}\|_2^2 \leq \|h - Ay_{(1:k)}\|_2^2$  and we're done). Then  $i \leq 2k$ , and we only need to bound the error  $\|h - A\hat{y}^i\|_2^2$  after time  $i$  where  $\hat{y}^i$  is the current OMP solution with support in  $\{1 \dots 2k\}$ , since any later  $y^i$  will have smaller error.

Let  $\Delta = \hat{y}^i - y'$ ,  $r^i = h - A\hat{y}^i$ , and  $r = h - Ay'$ . Then

$$\begin{aligned} \|A\Delta\|_2^2 &= r^\top A\Delta + (A\Delta - r)^\top A\Delta \\ &= r^\top A\Delta - (r^i)^\top A\Delta \\ &\leq \|r\|_2 \|A\Delta\|_2 + |(r^i)^\top A\Delta| \end{aligned}$$

by Cauchy-Schwarz. Using the fact  $x \leq b\sqrt{x} + c \Rightarrow x \leq (4/3)(b^2 + c)$  (which in turn can be checked using the quadratic formula and the fact  $2xy \leq x^2 + y^2$ ), the above inequality implies

$$\|A\Delta\|_2^2 \leq \frac{4}{3} (\|r\|_2^2 + |(r^i)^\top A\Delta|). \quad (1)$$

The selection criterion of OMP implies that if it choose a column  $j > 2k$  at time  $i$ , then

$$|a_j^\top r^i| \geq |a_\ell^\top r^i| \quad (\forall \ell \leq 2k). \quad (2)$$

Note that since  $\Delta$  has support  $\{1 \dots 2k\}$ ,  $A\Delta = A_{(1:2k)}\Delta$  where  $A_{(1:2k)}$  is the matrix  $A$  with zeros in all but the first  $2k$  columns. Therefore,

$$\begin{aligned} |(r^i)^\top A\Delta| &= |(r^i)^\top A_{(1:2k)}\Delta| \\ &\leq \|(r^i)^\top A_{(1:2k)}\|_\infty \|\Delta\|_1 \quad (\text{Cauchy-Schwarz}) \\ &\leq |a_j^\top r^i| \|\Delta\|_1 \quad (\text{ineq. (2)}) \\ &\leq (|a_j^\top r| + |a_j^\top A\Delta|) \|\Delta\|_1 \quad (\text{triangle ineq.}) \\ &\leq (|a_j^\top r| + \|a_j^\top A_{(1:2k)}\|_\infty \|\Delta\|_1) \|\Delta\|_1 \quad (\text{Cauchy-Sch.}) \\ &\leq |a_j^\top r| \|\Delta\|_1 + \mu(A) \|\Delta\|_1^2. \quad (\text{defn. coherence}) \end{aligned}$$

Therefore, continuing from ineq. (1), we have

$$\frac{3}{4} \|A\Delta\|_2^2 \leq \|r\|_2^2 + |a_j^\top r| \|\Delta\|_1 + \mu(A) \|\Delta\|_1^2. \quad (3)$$

Now we relate  $\|\Delta\|_1^2$  to  $\|A\Delta\|_2^2$ . Since  $\mu(A) \leq 1/10k$  and  $\Delta$  is  $2k$ -sparse, Lemma 6 (below) implies that  $\|A\Delta\|_2^2 \geq (1 - 0.2) \|\Delta\|_2^2$ . By Cauchy-Schwarz, we have

$$\|\Delta\|_1^2 \leq 2.5k \|A\Delta\|_2^2. \quad (4)$$

Combining ineq. (4) with ineq. (3) gives  $(3/4) \|A\Delta\|_2^2 \leq \|r\|_2^2 + |a_j^\top r| \|\Delta\|_1 + \mu(A) \|\Delta\|_1^2 \leq \|r\|_2^2 + |a_j^\top r| \|\Delta\|_1 + (1/4) \|A\Delta\|_1^2$ , which rearranges to become

$$\frac{1}{2} \|A\Delta\|_2^2 \leq \|r\|_2^2 + |a_j^\top r| \|\Delta\|_1. \quad (5)$$

The second term on the right side can be bounded using the fact  $xy \leq (x^2 + y^2)/2$ :

$$\begin{aligned} |a_j^\top r| \|\Delta\|_1 &\leq \frac{5k}{2} |a_j^\top r|^2 + \frac{\|\Delta\|_1^2}{10k} \\ &\leq \frac{5k}{2} |a_j^\top r|^2 + \frac{1}{4} \|A\Delta\|_2^2, \quad (\text{ineq. (4)}) \end{aligned}$$

whereupon

$$\frac{1}{4} \|A\Delta\|_2^2 \leq \|r\|_2^2 + 2.5k |a_j^\top r|^2. \quad (6)$$

Finally, we have  $|a_j^\top r| \leq |a_{i^*}^\top r|$ , where  $i^* \leq 2k$  was defined as the next column that OMP would select when the current solution is  $y'$  (i.e. when the residual is  $r = h - Ay'$ ). Since  $|a_{i^*}^\top r| \leq \|h - Ay_{(1:k)}\|_2^2/k$ , and also  $\|r\|_2^2 = \|h - Ay'\|_2^2 \leq \|h - Ay_{(1:k)}\|_2^2$ , we conclude from ineq. (6) that  $\|A\Delta\|_2^2 \leq 14 \|h - Ay_{(1:k)}\|_2^2$ , so

$$\begin{aligned} \|h - A\hat{y}^i\| &\leq \|A\Delta\|_2 + \|h - Ay'\|_2 \quad (\text{triangle ineq.}) \\ &\leq \sqrt{14} \|h - Ay_{(1:k)}\|_2 + \|h - Ay'\|_2 \\ &\leq (1 + \sqrt{14}) \|h - Ay_{(1:k)}\|_2. \end{aligned}$$

Squaring both sides gives the conclusion.  $\square$

**Lemma 6.** (Donoho et al., 2006) *If  $y \in \mathbb{R}^d$  is  $k$ -sparse and  $\mu(A) \leq \delta/(k - 1)$ , then  $\|Ay\|_2^2 \geq (1 - \delta) \|y\|_2^2$ .*

## C. Output Range of Induced Regression Problems

Lemma 8 implies that the output range of the regression problems induced by our reduction is bounded by  $O(\sqrt{k \log(m)/m})$ . Its proof is an easy consequence of the following concentration theorem.

**Lemma 7** (Eq. (2.35) in (Ledoux, 2001)). *Let  $\gamma$  be the standard  $D$ -dimensional Gaussian distribution  $N(0, I_D)$ . Then, for every  $C$ -Lipschitz function  $F : \mathbb{R}^D \rightarrow \mathbb{R}$  and every  $r \geq 0$ ,*

$$\gamma(\{F \geq \mathbb{E}F + r\}) \leq e^{-r^2/2C^2}.$$

To apply Lemma 7 to functions of matrices, the function should be Lipschitz with respect to the Frobenius norm  $\|\cdot\|_F$ .

**Lemma 8.** *Let  $\Theta \in \mathbb{R}^{m \times d}$ , where  $\Theta_{i,j}$  are i.i.d. standard Gaussian random variables, and let  $\mathcal{Y} = \{y \in \{0, 1\}^d : \|y\|_0 \leq k\}$ . Then  $\sup_{y \in \mathcal{Y}} \|\Theta y\|_\infty \leq O(\sqrt{k \ln m} + \sqrt{k \ln(1/\delta)})$  with probability at least  $1 - \delta$ .*

*Proof.* Let  $f(\Theta) = \sup_{y \in \mathcal{Y}} \|\Theta y\|_\infty$ . For  $J \subseteq \{1, \dots, d\} = [d]$ , denote by  $\Theta_J$  the matrix  $\Theta$  with zeros in all columns except those indicated by  $J$ . First, note that for any  $\Theta, \Theta' \in \mathbb{R}^{m \times d}$ , we have  $f(\Theta) - f(\Theta') \leq \sup_{y \in \mathcal{Y}} \|(\Theta - \Theta')y\|_\infty \leq \sup_{J \subset [d]: |J|=k} \|\Theta_J - \Theta'_J\|_\infty \leq \sqrt{k} \|\Theta - \Theta'\|_2 \leq \sqrt{k} \|\Theta - \Theta'\|_F$ , so  $f$  is  $\sqrt{k}$ -Lipschitz ( $\|\cdot\|_\infty$  is the induced matrix  $p$ -norm with  $p = \infty$ ). Next, to compute  $\mathbb{E}f(\Theta)$ , we use the fact that  $\sup_{y \in \mathcal{Y}} \|\Theta y\|_\infty = \sqrt{k} \max_{i=1, \dots, m} |Z_i|$ , where  $Z_1, \dots, Z_m$  are i.i.d.  $N(0, 1)$ . By Jensen's inequality,  $\mathbb{E} \max_i |Z_i| \leq \sqrt{\mathbb{E} \max_i Z_i^2}$ . Now consider  $\exp(t \mathbb{E} \max_i Z_i^2)$  for some  $t > 0$ . Again by Jensen's, this is at most  $\mathbb{E} \max_i e^{tZ_i^2} \leq \sum_i \mathbb{E} e^{tZ_i^2} = m/\sqrt{1-2t}$ , using the moment generating function of  $\chi^2$  random variables. So, we have  $\mathbb{E} \max_i Z_i^2 \leq \ln(m/\sqrt{1-2t})/t$ , which is at most  $(2.1) \ln m + 4$  for  $t = 0.49$ . Applying Lemma 7 concludes the proof.  $\square$

$(1+t)\epsilon \leq \exp(-(mdt^2/24)(\lambda/\lambda_1)) \leq \exp(-mt^2/24)$  (using the fact  $\lambda_1 \leq d\lambda$ ). This bound is  $\delta$  when  $t = \sqrt{(24/m) \ln(1/\delta)}$ .  $\square$

## D. Proof of Theorem 4

We make use of the following Chernoff bound for sums of  $\chi^2$  random variables (since the square of standard normal random variables are  $\chi^2$ -distributed with one degree of freedom), a proof of which can be found in the Appendix A of (Dasgupta, 2000).

**Lemma 9.** *Fix any  $\lambda_1 \geq \dots \geq \lambda_D > 0$ , and let  $X_1, \dots, X_D$  be i.i.d. random variables following the  $\chi^2$  distribution with one degree of freedom. Then*

$$\Pr \left[ \sum_{i=1}^D \lambda_i X_i > (1+t) \sum_{i=1}^D \lambda_i \right] \leq \exp \left( -\frac{Dt^2}{24} \cdot \frac{\lambda}{\lambda_1} \right)$$

for any  $0 < t < 1$ , where  $\lambda = (\lambda_1 + \dots + \lambda_D)/D$ .

*Proof of Theorem 4.* Write  $A = (1/\sqrt{m})[\theta_1 | \dots | \theta_m]^\top$ , where each  $\theta_i$  is an independent  $d$ -dimensional Gaussian random vector  $N(0, I_d)$ . Define  $v_x = B^\top x - \mathbb{E}[y|x]$  so  $\mathbb{E}_x \|v_x\|_2^2 = \epsilon$ , and assume without loss of generality that  $v_x$  has full  $d$ -dimensional support. Our goal is to show that with high probability,  $\mathbb{E}_x \|Av_x\|_2^2 \leq (1 + O(1/\sqrt{m}))\epsilon$ . Using this definition and linearity of expectation, we have  $\mathbb{E}_x \|Av_x\|_2^2 = (1/m) \mathbb{E}_x \sum_{i=1}^m (\theta_i^\top v_x)^2 = (1/m) \sum_{i=1}^m \theta_i^\top (\mathbb{E}_x v_x v_x^\top) \theta_i$ . Since  $N(0, I_d)$  is rotationally invariant and  $\mathbb{E}_x v_x v_x^\top$  is symmetric and positive definite, we can assume  $\mathbb{E}_x v_x v_x^\top$  is diagonal with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_d > 0$ . Then  $(1/m) \sum_{i=1}^m \theta_i^\top (\mathbb{E}_x v_x v_x^\top) \theta_i = (1/m) \sum_{i=1}^m \sum_{j=1}^d \lambda_j \theta_{ij}^2$ . Each  $\theta_{ij}^2$  is a  $\chi^2$  random variable with one degree of freedom, so  $\mathbb{E} \theta_{ij}^2 = 1$ . Thus, the expected value of the previous sum is  $\sum_{j=1}^d \text{trace}(\mathbb{E}_x v_x v_x^\top) = \mathbb{E}_x \text{trace}(v_x v_x^\top) = \mathbb{E}_x \|v_x\|_2^2$ . Now applying Lemma 9, with  $D = md$  variables and  $\lambda = (\lambda_1 + \dots + \lambda_d)/d$ , we have  $\Pr[(1/m) \sum_{i,j} \lambda_j \theta_{ij}^2 >$