Problem 1. In this problem, we will compute the shortest path between two vertices using linear programming. Suppose that we are given a directed graph $G = (V, E)$, two vertices $s, t \in V$, and a length function $\ell : E \to \mathbb{R}_{>0}$. Let $d(u, v)$ be the length of the shortest path between $u$ and $v$ in $G$ w.r.t. $\ell$. We introduce an LP variable $x_u$ for every vertex $u$ other than $s$. Consider the following linear program:

\begin{align*}
\text{maximize:} & \quad x_t \\
\text{subject to} & \quad x_v - x_u \leq \ell(u, v) \quad \text{for every edge } (u, v) \in E, \text{s.t. } u \neq s, v \neq s \\
& \quad x_v \leq \ell(s, v) \quad \text{for every edge } (s, v) \in E \\
& \quad x \geq 0
\end{align*}

Prove that the optimal LP value equals $d(s, t)$. Specifically, do the following.

- Prove that the optimal LP value is at least $d(s, t)$. To this end, construct a feasible solution whose value equals $d(s, t)$.

- Prove that the optimal LP value is at most $d(s, t)$.

Problem 2. Write the dual to the LP from Problem 1. Suppose that $P$ is a shortest path between $s$ and $t$. Construct a feasible solution to the dual whose value equals the length of $P$ (i.e. that equal $d(s, t)$).