Homework Assignment 2

TTIC 31010

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Problem 1. Let $G = (V, E)$ be an arbitrary directed graph, with a source $s$, a sink $t$, and a positive integer capacity $c(e)$ on every edge $e \in E$. Decide whether each of the following statements is true or false. If a statement is true, give a proof, and if it is false, show a counterexample.

- If all capacities $c(e)$ are even, then the value of the maximum flow is even.
- If all capacities $c(e)$ are odd, then the value of the maximum flow is odd.
- If $f$ is a maximum $s$-$t$ flow in $G$, then $f$ saturates every edge in $out(s)$ with flow. That is, for each $e \in out(s)$, $f(e) = c(e)$.

Problem 2. Consider the flow network $G$ shown in the figure below. For every edge $e$, its capacity $c(e)$ and its flow value $f(e)$ are written next to the edge ($f(e)$ appears in parentheses).

Part a. Is the flow $f$ a maximum flow in the graph? Prove your answer.
Part b. What is the maximum flow value? Prove your answer. 
Problem 3. Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and non-negative edge capacities $\{c(e)\}$. Let $f : E \to \mathbb{R}_{\geq 0}$ be a maximum flow in $G$. Let $G_f$ be the residual graph. Denote by $S$ the set of nodes reachable from $s$ in $G_f$ and by $T$ the set of nodes from which $t$ is reachable in $G_f$. That is,

\begin{align*}
S &= \{u : \text{there is a directed path from } s \text{ to } u \text{ in } G_f\}, \\
T &= \{v : \text{there is a directed path from } v \text{ to } t \text{ in } G_f\}.
\end{align*}

Prove that $V = S \cup T$ if and only if $G$ has a unique $s$-$t$ minimum cut (an $s$-$t$ cut whose capacity is strictly less than the capacity on any other $s$-$t$ cut).