Problem 1. Let $A$ be a linear operator from $\ell^d_2$ to $\ell^d_2$. Suppose that $A$ has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d > 0$. Compute the distortion of $A$. 

Problem 2. 

1. Suppose that $f$ is a linear map from $\ell^d_1$ to $\ell^d_2$. Prove that $f$ has distortion at least $\sqrt{d}$. 

   Hint: Consider the standard basis $e_1, \ldots, e_d$ of $\ell^d_1$. Let $r_1, \ldots, r_d \in \{\pm 1\}$ be independent unbiased Bernoulli random variables. Choose an index $j \in \{1, \ldots, d\}$ uniformly at random. Define $r'_i$ by $r'_i = r_i$ if $i \neq j$, and $r'_i = -r_i$ if $i = j$. (That is, $r'_i = r_i$ for all but one index $i$.) Let r.v. $u = \sum_{i=1}^d r_i e_i$ and $u' = \sum_{i=1}^d r'_i e_i$. Compute the value of 

   $$\frac{E[\|u - u'\|^2]}{E[\|u - (-u)\|^2]}$$

   and 

   $$\frac{E[\|f(u) - f(u')\|^2]}{E[\|f(u) - f(-u)\|^2]}.$$ 

2. Give an example of a linear map $f$ from $\ell^d_1$ to $\ell^d_2$ that has distortion $\sqrt{d}$. 

3*. (extra credit) Suppose that $f$ is a differentiable bijective map from $\ell^d_1$ to $\ell^d_2$ ($f$ is not necessarily linear). Prove that $f$ has distortion at least $\sqrt{d}$. 

Problem 3. 

1. Show that every embedding of the $n$-cycle into $\mathbb{R}$ has distortion $\Omega(n)$. 

2. Recall that $K_{3,3}$ is the complete bipartite graph with parts of size 3.
Let $G$ be the graph obtained from $K_{3,3}$ by replacing every edge with a path of length $n$. Show that every embedding of the shortest path metric on $G$ into the Euclidean plane, has distortion $\Omega(n)$.

3*. (extra credit) Prove that every metric space on $n$ points embeds into $\mathbb{R}$ with distortion $O(n)$.

**Problem 4* [extra credit].** Consider two metric spaces $(X, d_X)$ and $(Y, d_Y)$. Let $A \subset X$ be a subset of $X$ and $f : A \to Y$ be a Lipschitz map from $A$ to $Y$. We say that a map $\tilde{f} : X \to Y$ is an extension of $f$ if $\tilde{f}(x) = f(x)$ for every $x \in A$. The Lipschitz extendability constant $e_k(X, Y)$ is the infimum over all numbers $C$ such that the following property holds: for every $A \subset X$ of size at most $k$ and every map $f : A \to Y$ there exists an extension $\tilde{f} : X \to Y$ of $f$ with $\|\tilde{f}\|_{Lip} \leq C\|f\|_{Lip}$.

1. Give an example of two normed spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ such that $e_3(U, V) > 1$.

2. Prove that for every metric space $(X, d_X)$ and every $k$, $e_k(X, \mathbb{R}) = 1$.

3. Prove that for every metric space $(X, d_X)$, $k$ and $N$, $e_k(X, \ell_N^\infty) = 1$.

4. Prove that for every $X \subset \ell_2^2$, $e_k(X, \ell_2^2) = 1$.

You may assume that $X$ is a finite metric space in parts 2–4.