

Approximation Algorithm for Non-Boolean MAX k -CSP

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Abstract. In this paper, we present a randomized polynomial-time approximation algorithm for MAX k -CSP $_d$. In MAX k -CSP $_d$, we are given a set of predicates of arity k over an alphabet of size d . Our goal is to find an assignment that maximizes the number of satisfied constraints.

Our algorithm has approximation factor $\Omega(kd/d^k)$ (when $k \geq \Omega(\log d)$). This bound is asymptotically optimal assuming the Unique Games Conjecture. The best previously known algorithm has approximation factor $\Omega(k \log d/d^k)$.

We also give an approximation algorithm for the boolean MAX k -CSP $_2$ problem with a slightly improved approximation guarantee.

1 Introduction

We design an approximation algorithm for the MAX k -CSP $_d$, the maximum constraint satisfaction problem with k -ary predicates and domain size d . In this problem, we are given a set of variables $\{x_u\}_{u \in X}$ and a set of predicates \mathcal{P} . Each variable x_u takes values in $[d] = \{1, \dots, d\}$. Each predicate $P \in \mathcal{P}$ depends on at most k variables. Our goal is to assign values to variables so as to maximize the number of satisfied constraints.

There has been a lot of interest in finding the approximability of MAX k -CSP $_d$ in the complexity community motivated by the connection of MAX k -CSP $_d$ to k -bit PCPs. Let us briefly overview known results. Samorodnitsky and Trevisan [9] showed that the boolean MAX k -CSP $_2$ problem cannot be approximated within a factor of $\Omega(2^{2\sqrt{k}}/2^k)$ if $P \neq NP$. Later Engebretsen and Holmerin [5] improved this bound to $\Omega(2^{\sqrt{2k}}/2^k)$. For non-boolean MAX k -CSP $_d$, Engebretsen [4] proved a hardness result of $2^{O(\sqrt{d})}/d^k$. Much stronger inapproximability results were obtained assuming the Unique Games Conjecture (UGC). Samorodnitsky and Trevisan [10] proved the hardness of $O(k/2^k)$ for the boolean MAX k -CSP $_2$. Austrin and Mossel [1] and, independently, Guruswami and Raghavendra [6] proved the hardness of $O(kd^2/d^k)$ for non-boolean MAX k -CSP $_d$. Moreover, Austrin and Mossel [1] proved the hardness of $O(kd/d^k)$ for every d and infinitely many k ; specifically, their result holds for d and k such that $k = (d^t - 1)/(d - 1)$ for some $t \in \mathbb{N}$. Very recently, Håstad strengthened the result of Austrin and Mossel and showed the hardness of $O(kd/d^k)$ for every d and $k \geq d$ [private communication].

On the positive side, approximation algorithms for the problem have been developed in a series of papers by Trevisan [12], Hast [7], Charikar, Makarychev and Makarychev [3], and Guruswami and Raghavendra [6]. The best currently known algorithm for k -CSP $_d$ by Charikar et al [3] has approximation factor of $\Omega(k \log d/d^k)$. Note that a trivial algorithm for MAX k -CSP $_d$ that just picks a random assignment satisfies each constraint with probability at least $1/d^k$, and therefore its approximation ratio is $1/d^k$.

The problem is essentially settled in the boolean case. We know that the optimal approximation factor is $\Theta(k/2^k)$ assuming UGC. However, best known lower and upper bounds for the non-boolean

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case do not match. In this paper, we present an approximation algorithm for non-boolean MAX k -CSP $_d$ with approximation factor $\Omega(kd/d^k)$ (for $k \geq \Omega(\log d)$). This algorithm is asymptotically optimal assuming UGC — it is within a constant factor of the upper bounds of Austrin and Mossel and of Håstad (for k of the form $(d^t - 1)/(d - 1)$ and for $k \geq d$, respectively). Our result improves the best previously known approximation factor of $\Omega(k \log d/d^k)$.

Related work Raghavendra studied a more general MAX CSP(\mathcal{P}) problem [8]. He showed that the optimal approximation factor equals the integrality gap of the standard SDP relaxation for the problem (assuming UGC). His result applies in particular to MAX k -CSP $_d$. However, the SDP integrality gap of MAX k -CSP $_d$ is not known.

Overview We use semidefinite programming (SDP) to solve the problem. In our SDP relaxation, we have an “indicator vector” u_i for every variable x_u and value i ; we also have a “indicator vector” z_C for every constraint C . In the intended solution, u_i is equal to a fixed unit vector \mathbf{e} if $x_u = i$, and $u_i = 0$ if $x_u \neq i$; similarly, $z_C = \mathbf{e}$ if C is satisfied, and $z_C = 0$, otherwise.

It is interesting that the best previously known algorithm for the problem [3] did not use this SDP relaxation; rather it reduced the problem to a binary k -CSP problem, which it solved in turn using semidefinite programming. The only previously known algorithm [6] that directly rounded an SDP solution for MAX k -CSP $_d$ had approximation factor $\Omega\left(\frac{k/d^t}{d^k}\right)$.

One of the challenges of rounding the SDP solution is that vectors u_i might have different lengths. Consequently, we cannot just use a rounding scheme that projects vectors on a random direction and then chooses vectors that have largest projections, since this scheme will choose longer vectors with disproportionately large probabilities. To deal with this problem, we first develop a rounding scheme that rounds *uniform* SDP solutions, solutions in which all vectors are “short”. Then we construct a randomized reduction that converts any instance to an instance with a uniform SDP solution.

Our algorithm for the uniform case is very simple. First, we choose a random Gaussian vector g . Then for every u , we find u_i that has the largest projection on g (in absolute value), and let $x_u = i$. However, the analysis of this algorithm is quite different from analyses of similar algorithms for other problems: when we estimate the probability that a constraint C is satisfied, we have to analyze the correlation of all vectors u_i with vector z_C (where $\{u_i\}$ are SDP vectors for variables x_u that appear in C , z_C is the SDP vector for C), whereas the standard approach would be to look only at pairwise correlations of vectors $\{u_i\}$; this approach does not work in our case, however, since vectors corresponding to an assignment that satisfies C may have very small pairwise correlations, but vectors corresponding to assignments that do not satisfy C may have much larger pairwise correlations.

Remark 1.1. We study the problem only in the regime when $k \geq \Omega(\log d)$. In Theorem 5.1, we prove that when $k = O(\log d)$ our algorithm has approximation factor $e^{\Omega(k)}/d^k$. However, in this regime, a better approximation factor of $\Omega(d/d^k)$ can be obtained by a simple greedy approach.

Other Results We also apply our SDP rounding technique to the boolean MAX k -CSP Problem. We give an algorithm that has approximation guarantee $\approx 0.62 k/2^k$ for sufficiently large k . That slightly improves the best previously known guarantee of $\approx 0.44 k/2^k$ [3]. We present this result in Appendix B.

2 Preliminaries

We apply the approximation preserving reduction of Trevisan [12] to transform a general instance of MAX k -CSP $_d$ to an instance where each predicate is a conjunction of terms of the form $x_u = i$. The reduction replaces a predicate P , which depends on variables x_{v_1}, \dots, x_{v_k} , with a set of clauses

$$\{(x_{v_1} = i_1) \wedge \dots \wedge (x_{v_k} = i_k) : P(i_1, \dots, i_k) \text{ is true}\}.$$

Then it is sufficient to solve the obtained instance. We refer the reader to [12] for details. We assume below that each predicate is a clause of the form $(x_{v_1} = i_1) \wedge \dots \wedge (x_{v_k} = i_k)$.

Definition 2.1 (Constraint satisfaction problem). *An instance \mathcal{I} of MAX CSP $_d$ consists of*

- a set of “indices” X ,
- a set of variables $\{x_u\}_{u \in X}$ (there is one variable x_u for every index $u \in X$),
- a set of clauses \mathcal{C} .

Each variable x_u takes values in the domain $[d] = \{1, \dots, d\}$. Each clause $C \in \mathcal{C}$ is a set of pairs (u, i) where $u \in X$ and $i \in [d]$. An assignment $x_u = x_u^*$ satisfies a clause C if for every $(u, i) \in C$, we have $x_u^* = i$. We assume that no clause C in \mathcal{C} contains pairs (u, i) and (u, j) with $i \neq j$ (no assignment satisfies such clause). The length of a clause C is $|C|$. The support of C is $\text{supp}(C) = \{u : (u, i) \in C\}$.

The value of an assignment x_u^* is the number of constraints in \mathcal{C} satisfied by x_u^* . Our goal is to find an assignment of maximum value. We denote the value of an optimal assignment by $OPT = OPT(\mathcal{I})$.

In the MAX k -CSP $_d$ problem, we additionally require that all clauses in \mathcal{C} have length at most k .

We consider the following semidefinite programming (SDP) relaxation for MAX CSP $_d$. For every index $u \in X$ and $i \in [d]$, we have a vector variable u_i ; for every clause C , we have a vector variable z_C .

$$\begin{aligned} & \text{maximize: } \sum_{C \in \mathcal{C}} \|z_C\|^2 \\ & \text{subject to} \\ & \sum_{i=1}^d \|u_i\|^2 \leq 1 && \text{for every } u \in X \\ & \langle u_i, u_j \rangle = 0 && \text{for every } u \in X, i, j \in [d] \text{ (} i \neq j \text{)} \\ & \langle u_i, z_C \rangle = \|z_C\|^2 && \text{for every } C \in \mathcal{C}, (u, i) \in C \\ & \langle u_j, z_C \rangle = 0 && \text{for every } C \in \mathcal{C}, (u, i) \in C \text{ and } j \neq i \end{aligned}$$

Denote the optimal SDP value by $SDP = SDP(\mathcal{I})$. Consider the optimal solution x_u^* to an instance \mathcal{I} and the corresponding SDP solution defined as follows,

$$u_i = \begin{cases} \mathbf{e}, & \text{if } x_u^* = i; \\ 0, & \text{otherwise;} \end{cases} \quad z_C = \begin{cases} \mathbf{e}, & \text{if } C \text{ is satisfied;} \\ 0, & \text{otherwise;} \end{cases}$$

where \mathbf{e} is a fixed unit vector. It is easy to see that this is a feasible SDP solution and its value equals $OPT(\mathcal{I})$. Therefore, $SDP(\mathcal{I}) \geq OPT(\mathcal{I})$.

Definition 2.2. We say that an SDP solution is uniform if $\|u_i\|^2 \leq 1/d$ for every $u \in X$ and $i \in [d]$.

Definition 2.3. Let ξ be a standard Gaussian variable with mean 0 and variance 1. We denote

$$\Phi(t) = \Pr(|\xi| \leq t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^t e^{-x^2/2} dx, \text{ and}$$

$$\bar{\Phi}(t) = 1 - \Phi(t) = \Pr(|\xi| > t).$$

We will use the following lemma, which we prove in Appendix.

Lemma 2.1. For every $t > 0$ and $\beta \in (0, 1]$, we have

$$\bar{\Phi}(\beta t) \leq \bar{\Phi}(t)^{\beta^2}.$$

We will also use the following result of Šidák [11]:

Theorem 2.1 (Šidák [11]). Let ξ_1, \dots, ξ_r be Gaussian random variables with mean zero and an arbitrary covariance matrix. Then for any positive t_1, \dots, t_r ,

$$\Pr(|\xi_1| \leq t_1, |\xi_2| \leq t_2, \dots, |\xi_r| \leq t_r) \geq \prod_{i=1}^r \Pr(|\xi_i| \leq t_i).$$

3 Rounding Uniform SDP Solutions

In this section, we present a rounding scheme for uniform SDP solutions.

Lemma 3.1. There is a randomized polynomial-time algorithm that given an instance \mathcal{I} of the MAX CSP_d problem (with $d \geq 57$) and a uniform SDP solution, outputs an assignment x_u such that for every clause $C \in \mathcal{C}$:

$$\Pr(C \text{ is satisfied by } x_u) \geq \frac{\min(\|z_C\|^2 |C| d/8, e^{|C|})}{2d^{|C|}}.$$

Proof. We use the following rounding algorithm:

Rounding Scheme for Uniform SDP solutions

Input: an instance of the MAX CSP_d problem and a uniform SDP solution.

Output: an assignment $\{x_u\}$.

- Choose a random Gaussian vector g so that every component of g is distributed as a Gaussian variable with mean 0 and variance 1, and all components are independent.
- For every $u \in V$, let $x'_u = \arg \max_i |\langle u_i, g \rangle|$.
- For every $u \in V$, choose x''_u uniformly at random from $[d]$ (independently for different u).
- With probability 1/2 return assignment $\{x'_u\}$; with probability 1/2 return assignment $\{x''_u\}$.

For every clause C , let us estimate the probabilities that assignments x'_u and x''_u satisfy C . It is clear that x''_u satisfies C with probability $d^{-|C|}$. We prove now that x'_u satisfies C with probability at least $d^{-3|C|/4}$ if $\|z\|_C^2 \geq 8/(|C|d)$.

Claim. Suppose $C \in \mathcal{C}$ is a clause such that $\|z\|_C^2 \geq 8/(|C|d)$ and $d \geq 57$. Then the probability that the assignment x'_u satisfies C is at least $d^{-3|C|/4}$.

Proof. Denote $s = |C|$. We assume without loss of generality that for every $u \in \text{supp}(C)$, $(u, 1) \in C$. Note that for $(u, i) \in C$, we have $\|z_C\|^2 = \langle z_C, u_i \rangle \leq \|z_C\| \cdot \|u_i\| \leq \|z_C\|/\sqrt{d}$ (here we use that the SDP solution is uniform and therefore $\|u_i\|^2 \leq 1/d$). Thus $\|z_C\|^2 \leq 1/d$. In particular, $s = |C| \geq 8$ since $\|z\|_C^2 \geq 8/(|C|d)$.

For every $u \in \text{supp}(C)$, let $u_1^\perp = u_1 - z_C$. Let $\gamma_{u,1} = \langle g, u_1^\perp \rangle$ and $\gamma_{u,i} = \langle g, u_i \rangle$ for $i \geq 2$. Let $\gamma_C = \langle g, z_C \rangle$. All variables $\gamma_{u,i}, \gamma_C$ are Gaussian variables. Using that for every two vectors v and w , $\mathbb{E}[\langle g, v \rangle \cdot \langle g, w \rangle] = \langle v, w \rangle$, we get

$$\begin{aligned}\mathbb{E}[\gamma_C \cdot \gamma_{u,1}] &= \langle z_C, u_1 - z_C \rangle = \langle z_C, u_1 \rangle - \|z_C\|^2 = 0; \\ \mathbb{E}[\gamma_C \cdot \gamma_{u,i}] &= \langle z_C, u_i \rangle = 0 \quad \text{for } i \geq 2.\end{aligned}$$

Therefore, all variables $\gamma_{u,i}$ are independent from γ_C . (However, for $u', u'' \in \text{supp}(C)$ variables $\gamma_{u',i}$ and $\gamma_{u'',j}$ are not necessarily independent.) Let $M = \bar{\Phi}^{-1}(1/d^{s/2})/\sqrt{sd/8}$. We write the probability that x'_u satisfies C ,

$$\begin{aligned}\Pr(x'_u \text{ satisfies } C) &= \Pr(\arg \max_i |\langle g, u_i \rangle| = 1 \text{ for every } u \in \text{supp}(C)) \\ &= \Pr(|\langle g, u_1 \rangle| > |\langle g, u_i \rangle| \text{ for every } u \in \text{supp}(C), i \in \{2, \dots, d\}) \\ &= \Pr(|\gamma_{u,1} + \gamma_C| > |\gamma_{u,i}| \text{ for every } u \in \text{supp}(C), i \in \{2, \dots, d\}) \\ &\geq \Pr(|\gamma_{u,1}| \leq M/2, \text{ and } |\gamma_{u,i}| \leq M/2 \\ &\quad \text{for every } u \in \text{supp}(C), i \in \{2, \dots, d\} \mid |\gamma_C| > M) \cdot \Pr(|\gamma_C| > M).\end{aligned}$$

Since all variables $\gamma_{u,i}$ are independent from γ_C ,

$$\Pr(x'_u \text{ satisfies } C) \geq \Pr(|\gamma_{u,i}| \leq M/2 \text{ for every } u \in \text{supp}(C), i \in \{1, \dots, d\}) \cdot \Pr(|\gamma_C| > M).$$

By Šidák's Theorem (Theorem 2.1), we have

$$\Pr(x'_u \text{ satisfies } C) \geq \left(\prod_{u \in \text{supp}(C)} \prod_{i=1}^d \Pr(|\gamma_{u,i}| \leq M/2) \right) \cdot \Pr(|\gamma_C| > M). \quad (1)$$

We compute the variance of vectors $\gamma_{u,i}$. We use that $\text{Var}[\langle g, v \rangle] = \|v\|^2$ for every vector v and that the SDP solution is uniform.

$$\begin{aligned}\text{Var}[\gamma_{u,1}] &= \|u_1^\perp\|^2 = \|u_1 - z_C\|^2 = \|u_1\|^2 - 2\langle u_1, z_C \rangle + \|z_C\|^2 = \|u_1\|^2 - \|z_C\|^2 \leq \|u_1\|^2 \leq 1/d; \\ \text{Var}[\gamma_{u,i}] &= \|u_i\|^2 \leq 1/d \quad \text{for } i \geq 2.\end{aligned}$$

Hence since $\bar{\Phi}(t)$ is an increasing function and $\bar{\Phi}(\beta t) \leq \bar{\Phi}(t)^{\beta^2}$ (by Lemma 2.1), we have

$$\begin{aligned}\Pr(|\gamma_{u,i}| \leq M/2) &= \bar{\Phi}(M/(2\sqrt{\text{Var}[\gamma_{u,i}]}) \geq \bar{\Phi}(\sqrt{d}M/2) = 1 - \bar{\Phi}(\sqrt{d}M/2) \\ &\geq 1 - \bar{\Phi}(\sqrt{sd/8}M)^{2/s} = 1 - (d^{-s/2})^{2/s} = 1 - d^{-1}\end{aligned}$$

(recall that we defined M so that $\bar{\Phi}(\sqrt{sd/8}M) = d^{-s/2}$). Similarly, $\text{Var}[\gamma_C] = \|z_C\|^2 \geq 8/(sd)$ (by the condition of the lemma). We get (using the fact that $\bar{\Phi}(t)$ is a decreasing function),

$$\Pr(|\gamma_C| > M) = \bar{\Phi}(M/\sqrt{\text{Var}[\gamma_C]}) \geq \bar{\Phi}(M\sqrt{sd/8}) = d^{-s/2}.$$

Plugging in bounds for $\Pr(|\gamma_{u,i}| \leq M/2)$ and $\Pr(|\gamma_C| > M)$ into (1), we obtain

$$\Pr(x'_u \text{ satisfies } C) \geq (1 - d^{-1})^{ds} d^{-s/2} \geq d^{-3s/4}.$$

Here, we used that $(1 - d^{-1})^d \geq d^{-1/4}$ for $d \geq 57$ (the inequality $(1 - d^{-1})^d \geq d^{-1/4}$ holds for $d \geq 57$ since it holds for $d = 57$ and the left hand side, $(1 - d^{-1})^d$, is an increasing function, the right hand side, $d^{-1/4}$, is a decreasing function). \square

We conclude that if $\|z_C\|^2 \leq 8/(|C|d)$ then the algorithm chooses assignment x''_u with probability $1/2$ and this assignment satisfies C with probability at least $1/d^{|C|} \geq \|z_C\|^2 |C| d / (8 d^{|C|})$. So C is satisfied with probability at least, $1/d^{|C|} \geq \|z_C\|^2 |C| d / (16 d^{|C|})$; if $\|z_C\|^2 \geq 8/(|C|d)$ then the algorithm chooses assignment x' with probability $1/2$ and this assignment satisfies C with probability at least $d^{-3|C|/4} \geq e^{|C|}/d^{|C|}$ (since $e \leq 57^{1/4} \leq d^{1/4}$). In either case,

$$\Pr(C \text{ is satisfied}) \geq \frac{\min(\|z_C\|^2 |C| d / 8, e^{|C|})}{2d^{|C|}}.$$

\square

Remark 3.1. We note that we did not try to optimize all constants in the statement of Lemma 3.1. By choosing all parameters in our proof appropriately, it is possible to show that for every constant $\varepsilon > 0$, there is a randomized rounding scheme, $\delta > 0$ and d_0 such that for every instance of MAX CSP $_d$ with $d \geq d_0$ the probability that each clause C is satisfied is at least $\min((1 - \varepsilon)\|z_C\|^2 \cdot |C| d, \delta \cdot e^{\delta|C|})/d^{|C|}$.

4 Rounding Arbitrary SDP Solutions

In this section, we show how to round an arbitrary SDP solution.

Lemma 4.1. *There is a randomized polynomial-time algorithm that given an instance \mathcal{I} of the MAX CSP $_d$ problem (with $d \geq 113$) and an SDP solution, outputs an assignment x_u such that for every clause $C \in \mathcal{C}$:*

$$\Pr(C \text{ is satisfied by } x_u) \geq \frac{\min(\|z_C\|^2 |C| d / 64, 2e^{|C|/8})}{4d^{|C|}}.$$

Proof. For every index u , we sort all vectors u_i according to their length. Let S_u be the indices of $\lceil d/2 \rceil$ shortest vectors among u_i , and $L_u = [d] \setminus S_u$ be the indices of $\lfloor d/2 \rfloor$ longest vectors among u_i (we break ties arbitrarily). For every clause C let $r(C) = |\{(u, i) \in C : i \in S_u\}|$.

Claim. For every $i \in S_u$, we have $\|u_i\|^2 \leq 1/|S_u|$.

Proof. Let $i \in S_u$. Note that $\|u_i\|^2 + \sum_{j \in L_u} \|u_j\|^2 \leq 1$ (this follows from SDP constraints). There are at least $\lceil d/2 \rceil$ terms in the sum, and $\|u_i\|^2$ is the smallest among them (since $i \in S_u$). Thus $\|u_i\|^2 \leq 1/\lceil d/2 \rceil = 1/|S_u|$. \square

We use a combination of two rounding schemes: one of them works well on clauses C with $r(C) \geq |C|/4$, the other on clauses C with $r(C) \leq |C|/4$.

Lemma 4.2. *There is a polynomial-time randomized rounding algorithm that given an MAX CSP_d instance \mathcal{I} with $d \geq 113$ outputs an assignment x_u such that every clause C with $r(C) \geq |C|/4$ is satisfied with probability at least*

$$\frac{\min(\|z_C\|^2 |C| d/64, e^{|C|/4})}{2d^{|C|}}.$$

Proof. We will construct a sub-instance \mathcal{I}' with a uniform SDP solution and then solve \mathcal{I}' using Lemma 3.1. To this end, we first construct a partial assignment x_u . For every $u \in X$, with probability $|L_u|/d = \lfloor d/2 \rfloor / d$, we assign a value to x_u uniformly at random from L_u ; with probability $1 - |L_u|/d = |S_u|/d$, we do not assign any value to x_u . Let $A = \{u : x_u \text{ is assigned}\}$. Let us say that a clause C *survives* the partial assignment step if for every $(u, i) \in C$ either $u \in A$ and $i = x_u$, or $u \notin A$ and $i \in S_u$.

The probability that a clause C survives is

$$\begin{aligned} & \prod_{(u,i) \in C, i \in L_u} \Pr(x_u \text{ is assigned value } i) \prod_{(u,i) \in C, i \in S_u} \Pr(x_u \text{ is unassigned}) = \\ & \left(\frac{\lfloor d/2 \rfloor}{d} \cdot \frac{1}{\lfloor d/2 \rfloor} \right)^{|C| - r(C)} \cdot \left(\frac{\lceil d/2 \rceil}{d} \right)^{r(C)} = \frac{\lceil d/2 \rceil^{r(C)}}{d^{|C|}}. \end{aligned}$$

For every survived clause C , let $C' = \{(u, i) : u \notin A\}$. Note that for every $(u, i) \in C'$, we have $i \in S_u$. We get a sub-instance \mathcal{I}' of our problem on the set of unassigned variables $\{x_u : u \notin A\}$ with the set of clauses $\{C' : C \in \mathcal{C} \text{ survives}\}$. The length of each clause C' equals $r(C)$. In sub-instance \mathcal{I}' , we require that each variable x_u takes values in S_u . Thus \mathcal{I}' is an instance of MAX CSP_{d'} problem with $d' = |S_u| = \lceil d/2 \rceil$.

Now we transform the SDP solution for \mathcal{I} to an SDP solution for \mathcal{I}' : we let $z_{C'} = z_C$ for survived clauses C , remove vectors u_i for all $u \in A$, $i \in [d]$ and remove vectors z_C for non-survived clauses C . By Claim 4, this SDP solution is a uniform solution for \mathcal{I}' (i.e. $\|u_i\| \leq 1/d'$ for every $u \notin A$ and $i \in S_i$; note that \mathcal{I}' has alphabet size d'). We run the rounding algorithm from Lemma 3.1. The algorithm assigns values to unassigned variables x_u . For every *survived* clause C , we get

$$\begin{aligned} \Pr(C \text{ is satisfied by } x_u) &= \Pr(C' \text{ is satisfied by } x_u) \geq \frac{\min(\|z_C\|^2 |C'| d'/8, e^{|C'|})}{2d'^{|C'|}} \\ &= \frac{\min(\|z_C\|^2 r(C) d'/8, e^{r(C)})}{2d'^{r(C)}} \geq \frac{\min(\|z_C\|^2 |C| d/64, e^{|C|/4})}{2d^{r(C)}}. \end{aligned}$$

Therefore, for every clause C ,

$$\begin{aligned} \Pr(C \text{ is satisfied by } x_u) &\geq \Pr(C \text{ is satisfied by } x_u \mid C \text{ survives}) \Pr(C \text{ survives}) \\ &\geq \frac{\min(\|z_C\|^2 |C| d/64, e^{|C|/4})}{2d^{r(C)}} \times \frac{\lceil d/2 \rceil^{r(C)}}{d^{|C|}} \\ &= \frac{\min(\|z_C\|^2 |C| d/64, e^{|C|/4})}{2d^{|C|}}. \end{aligned}$$

□

Finally, we describe an algorithm for clauses C with $r(C) \leq |C|/4$.

Lemma 4.3. *There is a polynomial-time randomized rounding algorithm that given an MAX CSP_d instance \mathcal{I} outputs an assignment x_u such that every clause C with $r(C) \leq |C|/4$ is satisfied with probability at least $e^{|C|/8}/d^{|C|}$.*

Proof. We do the following independently for every vertex $u \in X$. With probability $3/4$, we choose x_u uniformly at random from L_u ; with probability $1/4$, we choose x_u uniformly at random from S_u . The probability that a clause C with $r(C) \leq |C|/4$ is satisfied equals

$$\begin{aligned} \prod_{(u,i) \in C, i \in L_u} \frac{3}{4|L_u|} \prod_{(u,i) \in C, i \in S_u} \frac{1}{4|S_u|} &= \frac{1}{d^{|C|}} \cdot \left(\frac{3d}{4|L_u|} \right)^{|C|-r(C)} \left(\frac{d}{4|S_u|} \right)^{r(C)} \\ &\geq \frac{1}{d^{|C|}} \cdot \left(\frac{3d}{4|L_u|} \right)^{3|C|/4} \left(\frac{d}{4|S_u|} \right)^{|C|/4} \geq \frac{1}{d^{|C|}} \cdot \left(\left(\frac{3}{2} \right)^{3/4} \left(\frac{d}{2(d+1)} \right)^{1/4} \right)^{|C|}. \end{aligned}$$

Note that $\left(\frac{3}{2}\right)^{3/4} \left(\frac{d}{2(d+1)}\right)^{1/4} \geq \left(\frac{3}{2}\right)^{3/4} \left(\frac{113}{2 \cdot 114}\right)^{1/4} \geq e^{1/8}$. Therefore, the probability that the clause is satisfied is at least $e^{|C|/8}/d^{|C|}$. \square

We run the algorithm from Lemma 4.2 with probability $1/2$ and the algorithm from Lemma 4.3 with probability $1/2$. Consider a clause $C \in \mathcal{C}$. If $r(C) \geq |C|/4$, we satisfy C with probability at least $\frac{\min(\|z_C\|^2|C|d/64, e^{|C|/4})}{4d^{|C|}}$. If $r(C) \leq |C|/4$, we satisfy C with probability at least $e^{|C|/8}/(2d^{|C|})$. So we satisfy every clause C with probability at least $\frac{\min(\|z_C\|^2|C|d/64, 2e^{|C|/8})}{4d^{|C|}}$. \square

5 Approximation Algorithm for MAX k -CSP_d

In this section, we present the main result of the paper.

Theorem 5.1. *There is a polynomial-time randomized approximation algorithm for MAX k -CSP_d that given an instance \mathcal{I} finds an assignment that satisfies at least $\Omega(\min(kd, e^{k/8})OPT(\mathcal{I})/d^k)$ clauses with constant probability.*

Proof. If $d \leq 113$, we run the algorithm of Charikar, Makarychev and Makarychev [3] and get $\Omega(k/d^k)$ approximation. So we assume below that $d \geq 113$. We also assume that $kd/d^k \geq 1/|\mathcal{C}|$, as otherwise we just choose one clause from \mathcal{C} and find an assignment that satisfies it. Thus d^k is polynomial in the size of the input.

We solve the SDP relaxation for the problem and run the rounding scheme from Lemma 4.1 d^k times. We output the best of the obtained solutions. By Lemma 4.1, each time we run the rounding scheme we get a solution with expected value at least

$$\begin{aligned} \sum_{C \in \mathcal{C}} \frac{\min(\|z_C\|^2|C|d/64, 2e^{|C|/8})}{4d^{|C|}} &\geq \sum_{C \in \mathcal{C}} \frac{\min(kd/64, 2e^{k/8})}{4d^k} \|z_C\|^2 \geq \frac{\min(kd/64, 2e^{k/8})}{4d^k} SDP(\mathcal{I}) \\ &\geq \frac{\min(kd/64, 2e^{k/8})}{4d^k} OPT(\mathcal{I}). \end{aligned}$$

Denote $\alpha = \frac{\min(kd/64, 2e^{k/8})}{4d^k}$. Let Z be the random variable equal to the number of satisfied clauses. Then $\mathbb{E}[Z] \geq \alpha OPT(\mathcal{I})$, and $Z \leq OPT(\mathcal{I})$ (always). Let $p = \Pr(Z \leq \alpha OPT(\mathcal{I})/2)$. Then

$$p \cdot (\alpha OPT(\mathcal{I})/2) + (1-p) \cdot OPT(\mathcal{I}) \geq \mathbb{E}[Z] \geq \alpha OPT(\mathcal{I}).$$

So $p \leq \frac{1-\alpha}{1-\alpha/2} = 1 - \frac{\alpha}{2-\alpha}$. So with probability at least $1 - p \geq \frac{\alpha}{2-\alpha}$, we find a solution of value at least $\alpha OPT(\mathcal{I})/2$ in one iteration. Since we perform $d^k > 1/\alpha$ iterations, we find a solution of value at least $\alpha OPT(\mathcal{I})/2$ with constant probability. \square

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A Proof of Lemma 2.1

In this section, we prove Lemma 2.1. We will use the following fact.

Lemma A.1. [see e.g. [2]] For every $t > 0$,

$$\frac{2t}{\sqrt{2\pi}(t^2+1)}e^{-\frac{t^2}{2}} < \bar{\Phi}(t) < \frac{2}{\sqrt{2\pi}t}e^{-\frac{t^2}{2}}.$$

Lemma 2.1. For every $t > 0$ and $\beta \in (0, 1]$, we have

$$\bar{\Phi}(\beta t) \leq \bar{\Phi}(t)^{\beta^2}.$$

Proof. Rewrite the inequality we need to prove as follows: $(\bar{\Phi}(\beta t))^{1/\beta^2} \leq \bar{\Phi}(t)$. Denote the left hand side by $f(\beta, t)$:

$$f(\beta, t) = \bar{\Phi}(\beta t)^{1/\beta^2}.$$

We show that for every $t > 0$, $f(\beta, t)$ is strictly increasing function as a function of $\beta \in (0, 1]$. Then,

$$(\bar{\Phi}(\beta t))^{1/\beta^2} = f(\beta) < f(1) = \bar{\Phi}(t).$$

We first prove that $\frac{\partial f(1,t)}{\partial \beta} > 0$. Write,

$$\frac{\partial f(1,t)}{\partial \beta} = -2\log(\bar{\Phi}(t))\bar{\Phi}(t) + t\bar{\Phi}'(t) = -2\log(\bar{\Phi}(t))\bar{\Phi}(t) - \frac{2te^{-t^2/2}}{\sqrt{2\pi}}.$$

Consider three cases. If $t \geq \sqrt{\frac{2e}{\pi}}$, then, by Lemma A.1,

$$\bar{\Phi}(t) < \frac{2}{\sqrt{2\pi t}}e^{-t^2/2} \leq e^{-1/2}e^{-t^2/2} = e^{-(t^2+1)/2}.$$

Hence, $-2\log(\bar{\Phi}(t)) > (t^2 + 1)$, and by Lemma A.1,

$$-2\log(\bar{\Phi}(t))\bar{\Phi}(t) > (t^2 + 1)\bar{\Phi}(t) > \frac{2te^{-t^2/2}}{\sqrt{2\pi}}.$$

If $t < \sqrt{\frac{2e}{\pi}}$, then let $\rho(x) = -\log x/(1-x)$ for $x \in (0, 1)$ and write,

$$-\log \bar{\Phi}(t) = \rho(\bar{\Phi}(t)) \cdot (1 - \bar{\Phi}(t)) = \frac{\rho(\bar{\Phi}(t))}{\sqrt{2\pi}} \int_{-t}^t e^{-x^2/2} dx \geq \frac{2\rho(\bar{\Phi}(t))te^{-t^2/2}}{\sqrt{2\pi}}.$$

Hence,

$$\frac{\partial f(1,t)}{\partial \beta} = -2\log(\bar{\Phi}(t))\bar{\Phi}(t) - \frac{2te^{-t^2/2}}{\sqrt{2\pi}} \geq \frac{2te^{-t^2/2}}{\sqrt{2\pi}} \times (2\rho(\bar{\Phi}(t))\bar{\Phi}(t) - 1).$$

For $x \in [1/3, 1]$, $2\rho(x)x > 1$, since the function $\rho(x)x$ is increasing and $\rho(1/3) > 3/2$. Hence $2\rho(\bar{\Phi}(t))\bar{\Phi}(t) > 1$, if $\bar{\Phi}(t) \geq 1/3$.

The remaining case is $t < \sqrt{\frac{2e}{\pi}}$ and $\bar{\Phi}(t) < 1/3$. Then, $\bar{\Phi}(t) \geq \bar{\Phi}(\sqrt{\frac{2e}{\pi}}) > 1/6$ and hence $\bar{\Phi}(t) \in (1/6, 1/3)$. Since the function $-x \log x$ is increasing on the interval $(0, e^{-1})$,

$$-2\log(\bar{\Phi}(t))\bar{\Phi}(t) > -2\log(1/6) \cdot \frac{1}{6} > \frac{1}{2}.$$

The function $te^{-t^2/2}$ attains its maximum at $t = 1$, thus

$$\frac{2te^{-t^2/2}}{\sqrt{2\pi}} \leq \frac{2e^{-1/2}}{\sqrt{2\pi}} < \frac{1}{2}.$$

We get

$$\frac{\partial f(1,t)}{\partial \beta} = -2\log(\bar{\Phi}(t))\bar{\Phi}(t) - \frac{2te^{-t^2/2}}{\sqrt{2\pi}} > 0.$$

Since $\frac{\partial f(1,t)}{\partial \beta} > 0$, for every $t > 0$, there exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, $f(1 - \varepsilon, t) < f(1, t)$. Particularly, for $t' = \beta t$,

$$f(\beta, t) = f(1, t')^{1/\beta^2} \geq f(1 - \varepsilon, t')^{1/\beta^2} = f((1 - \varepsilon)\beta, t).$$

□

B Improved Approximation Factor for Boolean Max k -CSP

In this section, we present an approximation algorithm for the boolean Maximum k -CSP problem, MAX k -CSP₂. The algorithm has approximation factor $0.626612 k/2^k$ if k is sufficiently large. This bound improves the previously best known bound of $0.44 k/2^k$ [3] (if k is sufficiently large).

Our algorithm is a slight modification of the algorithm for rounding uniform solutions of MAX k -CSP _{d} . We use the SDP relaxation presented in Section 2. Without loss of generality, we will assume below that all clauses have length exactly k . If a clause C is shorter, we can introduce $k - |C|$ new variables and append them to C . This transformation will not change the value of the instance.

First, we describe a rounding scheme for an SDP solution $\{u_1, u_2\}_{u \in X} \cup \{z_C\}_{C \in \mathcal{C}}$.

Lemma B.1. *There is a polynomial-time randomized rounding algorithm such that for every clause $C \in \mathcal{C}$ the probability that the algorithm satisfies C is at least*

$$\frac{1}{2^k \sqrt{2\pi k}} \int_0^\infty h_\beta(t)^k dt, \text{ where } h_\beta(t) = 2\Phi(\beta t) e^{-t^2/2},$$

and $\beta = \sqrt{k} \|z_C\|_2$.

Proof. We round the SDP solution as follows.

SDP Rounding Scheme for MAX k -CSP₂

Input: an instance of MAX k -CSP₂ and an SDP solution.

Output: an assignment $\{x_u\}$.

- Choose a random Gaussian vector g so that every component of g is distributed as a Gaussian variable with mean 0 and variance 1, and all components are independent.
- For every $u \in V$, let $x_u = \arg \max_i \langle u_i, g \rangle$.

Consider a clause $C \in \mathcal{C}$. We assume without loss of generality that $C = \{(u, 1) : u \in \text{supp}(C)\}$. Let $\gamma_C = \langle z_C, g \rangle$ and $\gamma_u = \langle u_2 - u_1 + z_C, g \rangle$ for $u \in \text{supp}(C)$. Note that for $u \in \text{supp}(C)$,

$$\text{Var}[\gamma_C] = \|z_C\|^2 = \beta^2/k,$$

$$\text{Var}[\gamma_u] = \|u_2 - u_1 + z_C\|^2 = \|u_1\|^2 + \|u_2\|^2 + \|z_C\|^2 - 2\langle u_1, z_C \rangle = \|u_1\|^2 + \|u_2\|^2 - \|z_C\|^2 \leq 1,$$

$$\mathbb{E}[\gamma_C \gamma_u] = \langle z_C, u_2 - u_1 + z_C \rangle = \langle z_C, u_2 \rangle - \langle z_C, u_1 \rangle + \langle z_C, z_C \rangle = 0 - \|z_C\|^2 + \|z_C\|^2 = 0.$$

Therefore, all random variables γ_u , for $u \in \text{supp} C$, are independent from γ_C . The probability that C is satisfied equals

$$\begin{aligned} \Pr(C \text{ is satisfied}) &= \Pr(\langle u_1, g \rangle > \langle u_2, g \rangle \text{ for every } u \in \text{supp}(C)) \\ &= \Pr(\gamma_C > \gamma_u \text{ for every } u \in \text{supp}(C)) \geq \Pr(|\gamma_u| < \gamma_C \text{ for every } u \in \text{supp}(C)) \\ &= \mathbb{E}_{\gamma_C} [\Pr(|\gamma_u| \leq \gamma_C \text{ for every } u \in \text{supp}(C) \mid \gamma_C)] \\ &\stackrel{\text{let } t=\gamma_C/\beta}{=} \frac{1}{\sqrt{2\pi k}} \int_{t=0}^\infty \Pr(|\gamma_u| \leq \beta t \text{ for every } u \in \text{supp}(C)) e^{-kt^2/2} dt. \end{aligned}$$

We use here that $\text{Var}[\gamma_C/\beta] = 1/k$. By Šidák's Theorem (Theorem 2.1), we have

$$\begin{aligned} \Pr(|\gamma_u| \leq \beta t \text{ for every } u \in \text{supp}(C)) &\geq \prod_{u \in \text{supp}(C)} \Pr(|\gamma_u| \leq \beta t) = \prod_{u \in \text{supp}(C)} \Phi(\beta t / \sqrt{\text{Var}[\gamma_u]}) \\ &\geq \prod_{u \in \text{supp}(C)} \Phi(\beta t) = \Phi(\beta t)^k. \end{aligned}$$

We conclude that

$$\Pr(C \text{ is satisfied}) \geq \frac{1}{2^k \sqrt{2\pi k}} \int_0^\infty h_\beta(t)^k dt.$$

□

Let $g(\beta) = \max_{t \in \mathbb{R}} h_\beta(t)$ ($h_\beta(t)$ attains its maximum since $h_\beta(t) \rightarrow 0$ as $t \rightarrow \infty$). Note that $g(\beta)$ is an increasing function since $h_\beta(t)$ is an increasing function of β for every fixed t . Additionally, $g(0) = 0$ and $\lim_{\beta \rightarrow \infty} g(\beta) = 2$ since $f(\beta, 1/\sqrt{\beta}) = 2\Phi(\sqrt{\beta})e^{-1/(2\beta)} \rightarrow 2$ as $\beta \rightarrow \infty$, and for every β and t , $f(\beta, t) \leq 2$. Therefore, g^{-1} is defined on $[0, 2)$. Let $\beta_0 = g^{-1}(1)$. It is easy to check numerically that $\beta_0 \in (1.263282, 1.263283)$.

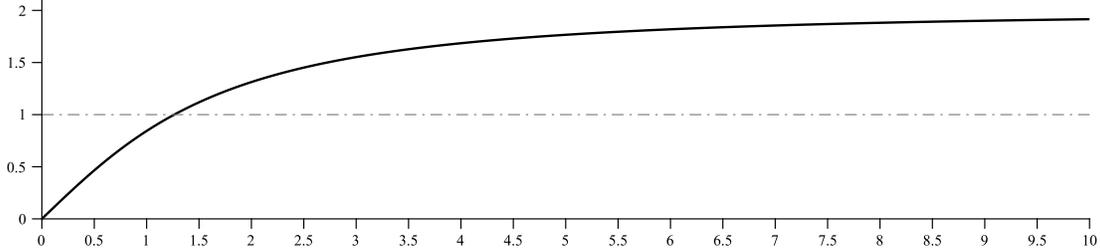


Fig. 1. The figure shows the graph of $g(t)$. We note that $g(t) > 1$ when $t > \beta_0 \approx 1.263282$.

Claim. For every $\beta > \beta_0$ there exists k_0 (which depends only on β) such that if $k \geq k_0$ and $\|z_C\| \geq \beta/\sqrt{k}$ then the probability that the algorithm from Lemma B.1 returns an assignment that satisfies C is at least $k^2/2^k$.

Proof. Let $\varepsilon_1 = (g(\beta) - 1)/2 > 0$. Let ε_2 be the measure of the set $\{t : h_\beta(t) > 1 + \varepsilon_1\}$. Since $h_\beta(t)$ is continuous, $\varepsilon_2 > 0$.

The probability that C is satisfied is at least

$$\frac{1}{2^k \sqrt{2\pi k}} \int_0^\infty h_\beta(t)^k dt \geq \frac{\varepsilon_2 (1 + \varepsilon_1)^k}{2^k \sqrt{2\pi k}}.$$

We choose k_0 so that for every $k \geq k_0$

$$\varepsilon_2 (1 + \varepsilon_1)^k \geq \sqrt{2\pi k} \cdot k^2.$$

Then if $k \geq k_0$ the probability that the clause is satisfied is at least $k^2/2^k$. □

Now we are ready to describe our algorithm.

Theorem B.1. *There is a randomized approximation algorithm for the boolean MAX k -CSP problem with approximation guarantee $\alpha_k k/2^k$ where $\alpha_k \rightarrow \alpha_0 \geq 0.626612$ as $k \rightarrow \infty$ and $\alpha_0 = 1/\beta_0^2$. (Here, as above, β_0 is the solution of the equation $g(\beta) = 1$ where $g(\beta) = \max_{t \in \mathbb{R}} 2\Phi(\beta t)e^{-t^2/2}$.)*

Proof. The algorithm with probability $p = 1/k$ rounds the SDP solution as described in Lemma B.1, with probability $1 - p$, it choses a completely random solution.

Let $\alpha < \alpha_0$. We will show that if k is large enough, every clause is satisfied with probability at least $\alpha k/2^k$. Let $\beta = (\beta_0 + \alpha^{-1/2})/2 \in (\beta_0, \alpha^{-1/2})$. Let k_0 be as in Claim B. Suppose that $k \geq \max(k_0, (1 - \alpha\beta^2)^{-1})$.

Consider a clause C . We show that the algorithm satisfies C with probability at least $\frac{\alpha \|z_C\|^2 k}{2^k}$. Indeed, we have:

- If $\|z_C\| < \beta/\sqrt{k}$, the clause is satisfied with probability at least $(1 - p)/2^k \geq \frac{(1-p)k \|z_C\|^2}{\beta^2 2^k} \geq \alpha \|z_C\|^2 k/2^k$.
- If $\|z_C\| \geq \beta/\sqrt{k}$, the clause is satisfied with probability at least $p \cdot k^2/2^k = k/2^k \geq k \|z_C\|^2/2^k$.

We conclude that the algorithm finds a solution that satisfies at least

$$\frac{\alpha k}{2^k} \sum_{C \in \mathcal{C}} \|z_C\|^2 = \frac{\alpha k}{2^k} \cdot SDP \geq \frac{\alpha k}{2^k} \cdot OPT$$

clauses in expectation. By running this algorithm polynomially many times (as we do in Theorem 5.1) we can find a solution of value at least $\alpha' k OPT/2^k$ for every constant $\alpha' < \alpha$ w.h.p. \square