1 Gram-Schmidt Orthonormalization

Exercise 1.1 If $U$ is unitary and $\lambda$ is a (complex) eigenvalue, prove that $|\lambda| = 1$.

Recall that $b_1, \ldots, b_k$ form an orthonormal basis if

$$\langle b_i, b_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Also it is easy to check that $b_1, \ldots, b_k$ are such that $\langle b_i, b_j \rangle = 0$ for $i \neq j$ then $b_1, \ldots b_k$ are linearly independent.

Theorem 1.2 (Gram-Schmidt Orthogonalization) If $v_1, \ldots, v_k$ are linearly independent, then there exist vectors $b_1, \ldots, b_k$ such that

1. $b_1, \ldots, b_k$ form an orthonormal basis.
2. For all $i \leq k$, Span$(b_1, \ldots, b_i) = \text{Span}(v_1, \ldots, v_k)$.

Proof: The proof is algorithmic and we claim the following process constructs $b_1, \ldots, b_k$ with the above property:

- Start with $b_1 = \frac{v_1}{\|v_1\|}$.
- For each $i = 2, \ldots, k$, set

$$v'_i = v_i - \langle v_i, b_1 \rangle \cdot b_1 - \cdots - \langle v_i, b_{i-1} \rangle \cdot b_{i-1} \quad \text{and} \quad b'_i = \frac{v'_i}{\|v'_i\|}$$

Prove by induction on $i$ that the vectors $b_1, \ldots, b_k$ satisfy the above properties. 

2 The Spectral Theorem

We can use Gram-Schmidt orthogonalization to prove the following theorem for every square matrix $A$. The spectral theorem then follows as an easy corollary. Recall that the characteristic polynomial of any matrix $A \in M_n(\mathbb{C})$ has $n$ complex roots.
Theorem 2.1 (Schur’s Theorem) Let $A \in \mathbb{C}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then there exists a unitary matrix $U$ and an upper-triangular matrix $T$ such that $A = UTU^*$. Also, the diagonal entries of $T$ are equal to $\lambda_1, \ldots, \lambda_n$.

Proof Sketch: $A$ must have at least one eigenvector corresponding to the eigenvalue $\lambda_1$. Consider $u_1$ such that $Au_1 = \lambda_1 u_1$ and $\|u_1\| = 1$. Complete it to a basis $u_1, v_2, \ldots, v_n$ for $\mathbb{C}^n$ and use Gram-Schmidt to obtain an orthonormal basis. Note that the first element of basis given by Gram-Schmidt will still be $u_1$. Let the orthonormal basis be $u_1, \ldots, u_n$ and consider a unitary matrix $U_1$ with columns $u_1, \ldots, u_n$. Then $U_1^* AU_1$ must be of the form

$$U_1^* AU_1 = \begin{pmatrix} \lambda_1 & B_1 \\ 0 & A_1 \end{pmatrix}$$

Since $A \sim U_1^* AU_1$, the characteristic polynomial of $U_1^* AU_1$ must be the same as that of $A$. Also, since $\det(tI - U_1^* AU_1) = (t - \lambda_1) \cdot \det(tI - A_1)$, $A_1$ must have eigenvalues $\lambda_2, \ldots, \lambda_n$. As above, we can find a unitary matrix $V_2 \in M_{n-1}(\mathbb{C})$ such that

$$V_2^* A_1 V_2 = \begin{pmatrix} \lambda_2 & B_2 \\ 0 & A_2 \end{pmatrix}$$

Take $U_2$ to be the matrix

$$U_2 = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & V_2 \end{pmatrix},$$

and note that $U_2^* U_1^* AU_1 U_2$ is of the form

$$U_2^* U_1^* AU_1 U_2 = \begin{pmatrix} \lambda_1 & * & B_3 \\ 0 & \lambda_2 & \vdots \\ 0 & \cdots & A_3 \end{pmatrix}.$$

Continuing this process, we get unitary matrices $U_1, \ldots, U_n$ such that

$$U_n^* \cdots U_1^* AU_1 \cdots U_n = T,$$

where $T$ is upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$. Taking $U = U_1 \cdots U_n$ proves the theorem.

A matrix $A$ is called normal if $AA^* = A^* A$. Schur’s theorem gives the spectral theorem for normal matrices as an easy corollary.
Corollary 2.2 (Spectral Theorem) Let $A$ be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\exists U$ such that $A = UDU^*$ where $D$ is a diagonal with entries $\lambda_1, \ldots, \lambda_n$.

**Proof:** Assuming Schur’s theorem, proceed in following steps.

1. If $A = UTU^*$ as in Schur’s theorem, then $TT^* = T^*T$.

2. Show that if $T$ is a triangular matrix which is normal, then $T$ must be diagonal.

Example 2.3 Suppose $A$ is normal and thus $A = UDU^*$. Let $U = [u_1, \ldots, u_n], D_{ii} = \lambda_i$. Then $Au_i = \lambda_i u_i$, and the $u_i$’s form an orthonormal basis of eigenvectors.

Exercise 2.4 Show that $A$ is diagonalizable iff for all eigenvalues, algebraic multiplicity = geometric multiplicity.

Exercise 2.5 $A = UDU^* \Rightarrow A = \sum_{i=1}^{n} \lambda_i u_i u_i^*$.

Definition 2.6 (Hermitian and Symmetric Matrices) An $n \times n$ matrix $A$ is Hermitian if $A^* = A$. $A$ is called symmetric if $A^T = A$.

Note that a Hermitian matrix is normal. Also, a real and symmetric matrix is Hermitian. Using the Spectral Theorem gives the following important conclusion for Hermitian matrices.

Proposition 2.7 Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Then all eigenvalues of $A$ must be real.

**Proof:** If $A$ is Hermitian, then using $A = UDU^*$ and $A = A^*$ gives $D = D^*$. Since $D$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$, we get that for each $i \in [n]$, $\lambda_i = \overline{\lambda_i}$, which means that all eigenvalues must be real.

Also, note that for a real symmetric matrix $A$, all the eigenvalues of $A$ are real and the proof of Schur’s theorem can be carried out over $\mathbb{R}^n$. This gives that a real symmetric matrix can be written as

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T,$$

where $\lambda_1, \ldots, \lambda_n$ are real eigenvalues and $u_1, \ldots, u_n \in \mathbb{R}^n$ form an orthonormal basis.

3 Adjacency matrices of graphs

Definition 3.1 (Adjacency Matrix) Let $G = (V, E)$ be a graph. Then the adjacency matrix $A$ is defined as

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

which is symmetric if $G$ is undirected.
If $A$ is the adjacency matrix of an undirected graph $G$ then all eigenvalues of $A$ are real. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues sorted so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. As before, we can find vectors $u_1, \ldots, u_n \in \mathbb{R}^n$ forming an orthonormal basis such that

$$A = \sum_{i=1}^{n} \mu_i u_i u_i^T.$$ 

Note that this implies $Au_i = \mu_i u_i$ and thus $u_1, \ldots, u_n$ are orthonormal (real) eigenvectors corresponding to (real) eigenvalues $\mu_1, \ldots, \mu_n$. 