Suggested Exercises

This is a list of practice problems that will periodically updated. Please note that you do not need to submit the solutions to these problems.

Discrete Probability

1. **An improved Schwartz-Zippel lemma.** Prove the following version of the Schwartz-Zippel lemma. Let \( f(x_1, \ldots, x_n) \) be a non-zero polynomial over a field \( F \) with degree-sequence \((d_1, \ldots, d_n)\), defined as follows: let \( d_1 \) be the maximum exponent of \( x_1 \) in \( f \) and let \( f_1(x_2, \ldots, x_n) \) be the coefficient of \( x_1^{d_1} \) in \( f \); then, let \( d_2 \) be the maximum exponent of \( x_2 \) in \( f_1 \) and let \( f_2(x_2, \ldots, x_n) \) be the coefficient of \( x_2^{d_2} \) in \( f_1 \) and so on. Suppose each variable \( x_i \) is assigned a value \( v_i \) chosen randomly and independently from a set \( S_i \subseteq F \). Then prove that

\[
P\left[f(v_1, \ldots, v_n) = 0\right] \leq \frac{d_1}{|S_1|} + \cdots + \frac{d_n}{|S_n|}.
\]

Can you construct the example of a polynomial where the above bound is strictly better than the bound we derived in the class?

2. **Patterns in coin tosses.** Consider an infinite sequence of independent tosses of a fair coin. Define the following random variables:

\[
Y_1 = \text{Number of occurrences of the pattern HTT in the first } n \text{ tosses}
\]

\[
Y_2 = \text{Number of occurrences of the pattern HTH in the first } n \text{ tosses}
\]

As discussed in class, it is easy to compute \( \mathbb{E}[Y_1] \) and \( \mathbb{E}[Y_2] \) and verify that they are equal. Now consider the following two random variables:

\[
Z_1 = \text{Number of tosses after which the pattern HTT first appears}
\]

\[
Z_2 = \text{Number of tosses after which the pattern HTH first appears}
\]

Compute \( \mathbb{E}[Z_1] \) and \( \mathbb{E}[Z_2] \) and verify that they are not equal. Why is one pattern more likely to occur first even though they are both occur an equal number of times (in expectation) in a given number of tosses?

3. **Random Polynomials.** For a prime number \( p \), the field \( \mathbb{F}_p \) has the elements \( \{0, 1, \ldots, p-1\} \), with addition and multiplication done modulo \( p \). A degree-\( d \) polynomial in the variable \( x \) over the field \( \mathbb{F}_p \) (for prime \( p \)) is defined as:

\[
P(x) = c_0 + c_1 \cdot x + \ldots + c_d \cdot x^d,
\]

where the coefficients \( c_0, \ldots, c_d \), and the variable \( x \) all take values in \( \mathbb{F}_p \), and all addition and multiplication is done modulo \( p \). A value \( x \in \mathbb{F}_p \) is called a root of \( P \) if \( P(x) = 0 \). Consider
picking a random polynomial $P$ by selecting $c_0, \ldots, c_d$ randomly from $\mathbb{F}_p$, and define the random variable

$$Z = \text{Number of roots of } P.$$ 

Calculate $\mathbb{E}[Z]$ and $\text{Var}[Z]$.

4. **Random Permutations.** Consider picking a permutation $\pi : [n] \to [n]$ uniformly at random (here $[n]$ denotes the set $\{1, \ldots, n\}$). A number $i \in [n]$ is said to be a fixed point of $\pi$ if $\pi(i) = i$. Define the random variable

$$Z_1 = \text{Number of fixed points of } \pi.$$ 

Compute $\mathbb{E}[Z_1]$ and $\text{Var}[Z_1]$. Also, recall that each permutation can be decomposed into cycles, obtained by looking at the orbits of the elements in $[n]$. For example, if $\pi(1) = 3$, $\pi(3) = 6$ and $\pi(6) = 1$, this gives a cycle of size 3. Define the random variable

$$Z_2 = \text{Number of cycles in } \pi.$$ 

Compute $\mathbb{E}[Z]$.

5. **Coupon Collection Revisited.** Recall that in class we showed that in the coupon collection problem, if $T$ is defined to be the time to collect coupons of all $n$ times, then $\mathbb{E}[T] = n \ln n + \Theta(n)$. Can you compute $\text{Var}[T]$? Use this to derive a bound on the probability that $T$ is significantly larger than $\mathbb{E}[T]$?