TTIC 106: Problem Set 4

Due Friday, March 12

Throughout this problem set let V be a vector space. We let x, y, z range over vectors in V nd α, β, δ range over dual vectors.

1.

Given a basis $x_1, \ldots, x_d$ for V show that there exists a unique basis $\alpha_1, \vdots, \alpha_d$ satisfying the following.

$$\alpha_j^T x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Also show that for any basis $\alpha_1, \vdots, \alpha_d$ there exists a unique basis $x_1, \ldots, x_d$ satisfying this condition.

2. For any vector y and basis $x_1, \ldots, x_d$, let $a_1(y), \ldots, a_d(y)$ be the corrdinates of y relative to this basis. These corrdinate are defined by the following condition.

$$y = a_1(y)x_1 + \cdots + a_d(y)x_d$$

Let β be a dual vector and let $b_1(\beta), \ldots, b_d(\beta)$ be the coordinates of β relative to the dual basis corresponding to the primal basis as defined in problem 1. Show the following.

$$\beta^T y = b_1(\beta)a_1(y) + \cdots + b_d(\beta)a_d(y)$$

In other words, given a basis a dual vector can be combined with a primal vector by taking the coordinate dot product between the coordinate sequences defined for the two vectors.

3. A function $M : V \times V \to \mathbb{R}$ is called bilinear if it satisfies the following conditions where we write $M(x, y)$ as xMy.

- $xM(ay + bz) = a(xMy) + b(xMz)$
- $(ax + bz)My = a(xMy) + b(zMy)$

Fix a basis $z_1, \ldots, z_d$ and let $a_1(x), \ldots, a_d(x)$ be the coordinates of x relative to that basis. Let $M_{i,j}$ be defined as follows.

$$M_{i,j} = z_i M z_j$$

Show the following.

$$xMy = \sum_{i=1}^{d} \sum_{j=1}^{d} a_i(x) M_{i,j} a_j(y)$$

4. An inner product operation on V is a bilinear operator $Q : V \times V \to \mathbb{R}$ satisfying the following conditions where we write $Q(x, y)$ as xQy.

- $xQy = yQx$
- $xQx > 0$ for $x \neq 0$

Show that for any inner product operation Q there exists a basis such that we have the following where $a_1(x)$, ..., $a_d(x)$ are the coordinates of x in that basis.

$$xQy = a_1(x)a_1(y) + \cdots + a_d(x)a_d(y)$$

. In other words for any inner product operation Q there exists a basis where Q is represented by the identity matrix (relative to that basis).

5. Let Q and W be two different inner product operations on V. Define a nontrivial linear operator of type $V \to V$ using Q qnd W. (For example, Q might be the "ambient" inner product and W might be Σ^{-1} for a covariance matrix Σ).