CS 39600: Theory of Metric Embeddings **Homework 2**
Date: Feb 2, 2006 **Due:** Feb 9, 2006

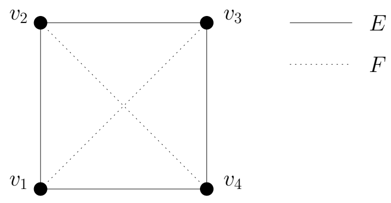

Figure 1: The 4-cycle C_4 with node set $V = \{v_1, v_2, v_3, v_4\}$.

1. Consider the 4-cycle C_4 depicted in Figure 1. Let V denote the node-set of C_4, and let d denote the shortest path metric. Further, let $E \subset V \times V$ denote the set of edges, and let $F \subset V \times V$ denote the set of diagonals. Define

$$d^2(E) := \sum_{(x,y)\in E} (d(x,y))^2 \text{ and } d^2(F) := \sum_{(x,y)\in F} (d(x,y))^2 \ .$$

Consider the ratio $R_{E,F}(d) = \sqrt{d^2(F)/d^2(E)}$. Assume that you have an embedding into ℓ_2 and let ρ denote the metric on V induced by this embedding.

(a) Show that if the distortion of the embedding is D, then $R_{E,F}(\rho) \geq \frac{1}{D} R_{E,F}(d)$.

(b) Let x_1, x_2, x_3, x_4 be arbitrary points in ℓ_2 for some dimension. Show that $\|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 \leq \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_1\|^2$.

(c) Conclude that the minimum distortion for embedding the 4-cycle into ℓ_2 is $\sqrt{2}$.

2. Let H_k denote the hypercube of dimension k, i.e., a graph with vertex set $V = \{0,1\}^k$ and edge-set $E = \{(x,y) \in V \times V \mid x \text{ and } y \text{ differ in exactly one bit}\}$. Let d denote the shortest-path metric on H_k. Let F denote the set of antipodal pairs (i.e. pairs of nodes that are exactly at distance k).

(a) Show that $R_{E,F}(d) = \sqrt{k}$.

(b) Show that for any embedding of H_k into ℓ_2 the corresponding metric ρ has $R_{E,F}(\rho) \leq 1 \Leftrightarrow \rho^2(F) \leq \rho^2(E)$. Use induction over k (note that $C_4 = H_2$; you will need exercise 1(b) in the induction step).

(c) Conclude that there exist squared ℓ_2-metrics (metrics from $\mathcal{L}_2$) that require distortion $\sqrt{\log n}$ for embedding into ℓ_2.