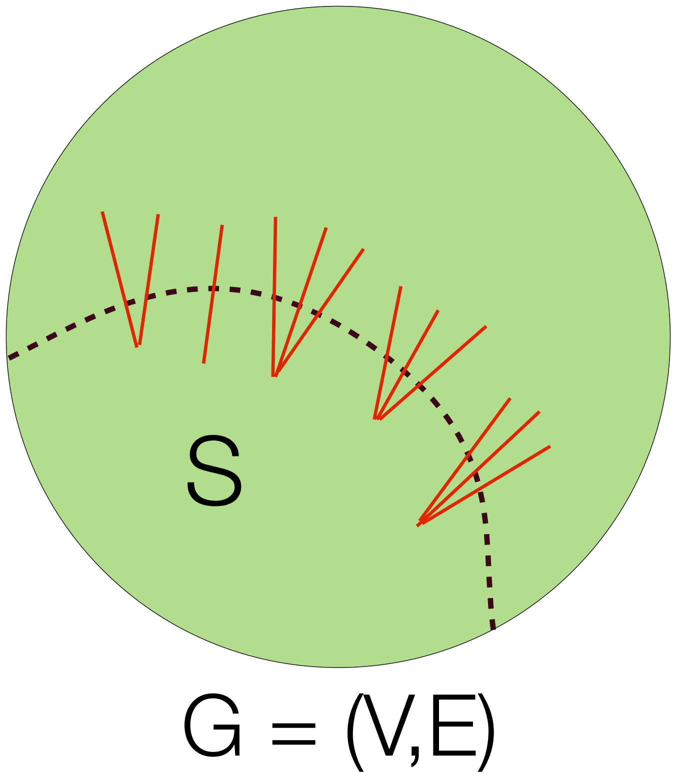
# Reductions Between Expansion Problems

Prasad Raghavendra
Georgia Tech.

David Steurer
MSR New England

Madhur Tulsiani
Princeton University

# Graph Expansion




d-regular

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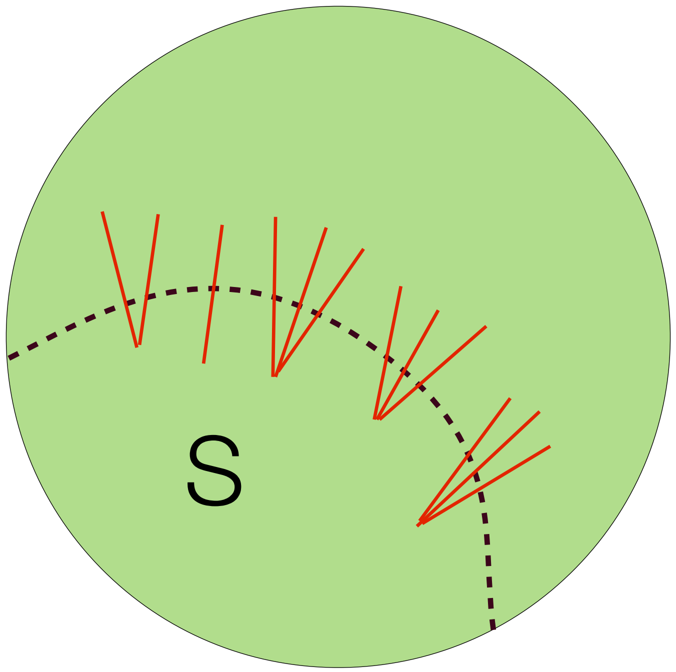


G = (V,E)

d-regular

$$\Phi_G(S) = \frac{E(S, V \setminus S)}{d|S|}$$

$$= \frac{\mathbf{P}_{(x,y) \in E}(x \in S,\ y \notin S)}{\mathbf{P}_{(x,y) \in E}(x \in S)}$$

$$\Phi_G = \min_{|S| \leq n/2} \{\Phi_G(S)\}$$

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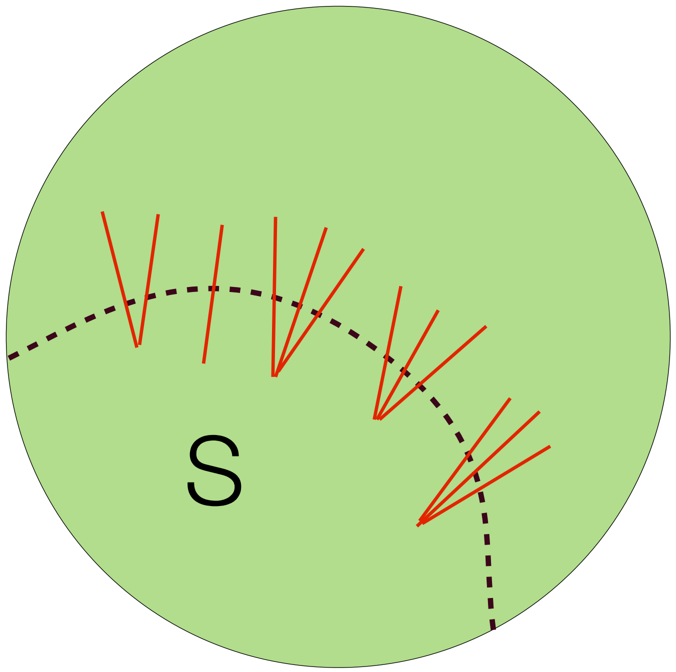


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- Expansion measures the probability of a random edge crossing a set S.
- Approximating the expansion of a graph is important for algorithms and also a fundamental problem in complexity.

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- Define the measure of a set by the fraction of edges landing in it.

$$\mu(S) = \mathbf{P}_{(x,y) \in E}(x \in S)$$
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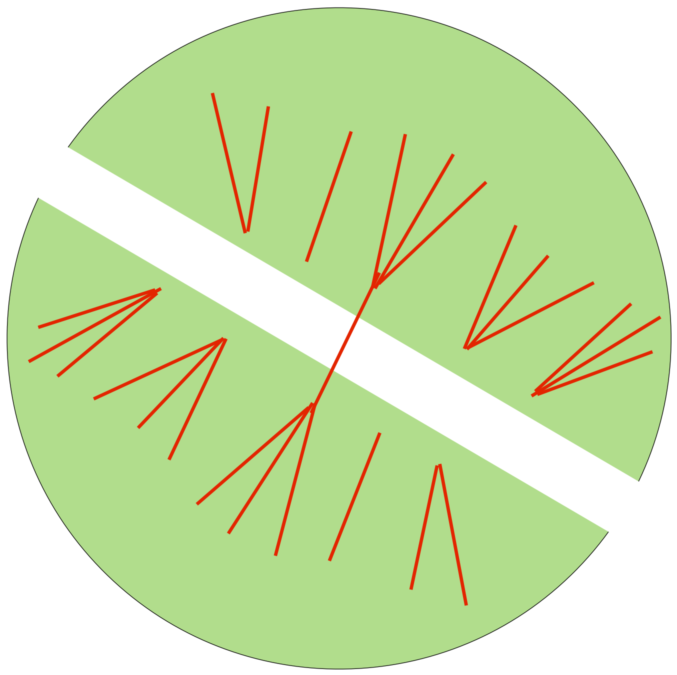
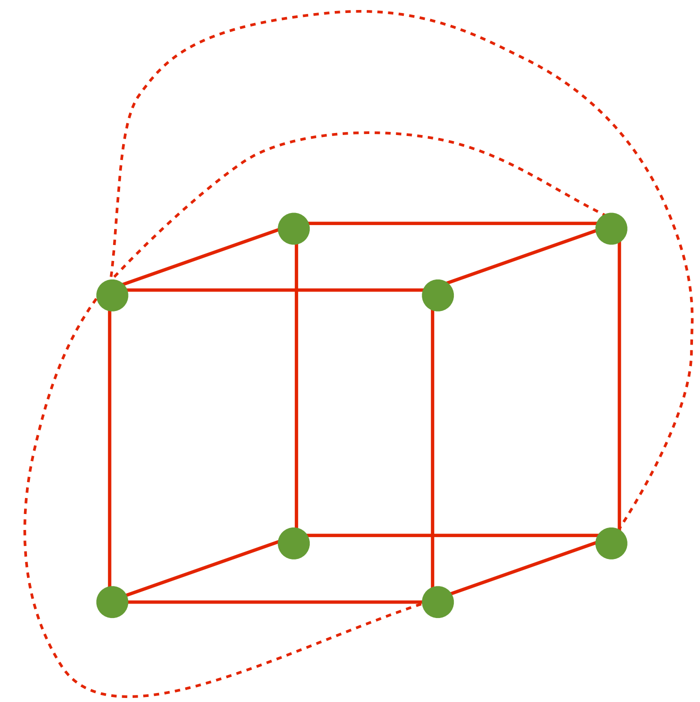
- The expansion profile of a graph may look very different at different scales.



# An example: the noise graph on $\{0,1\}^n$



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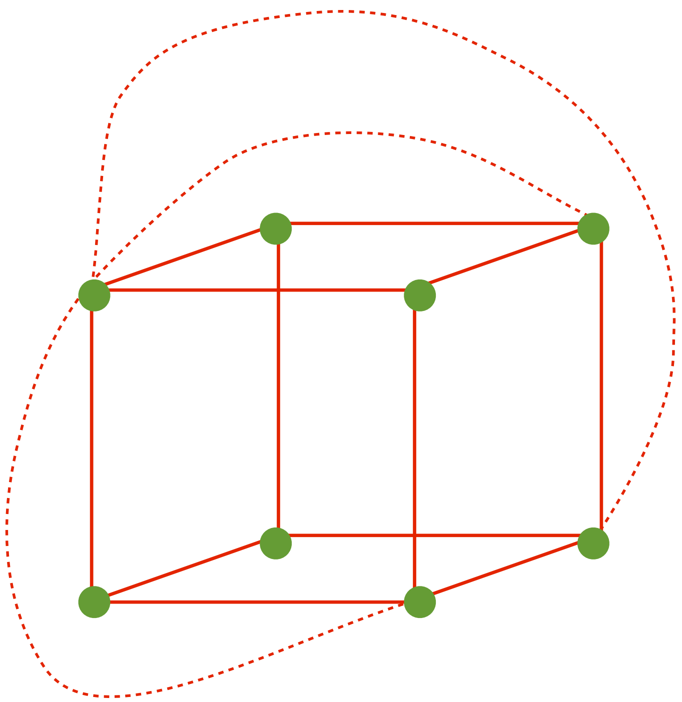

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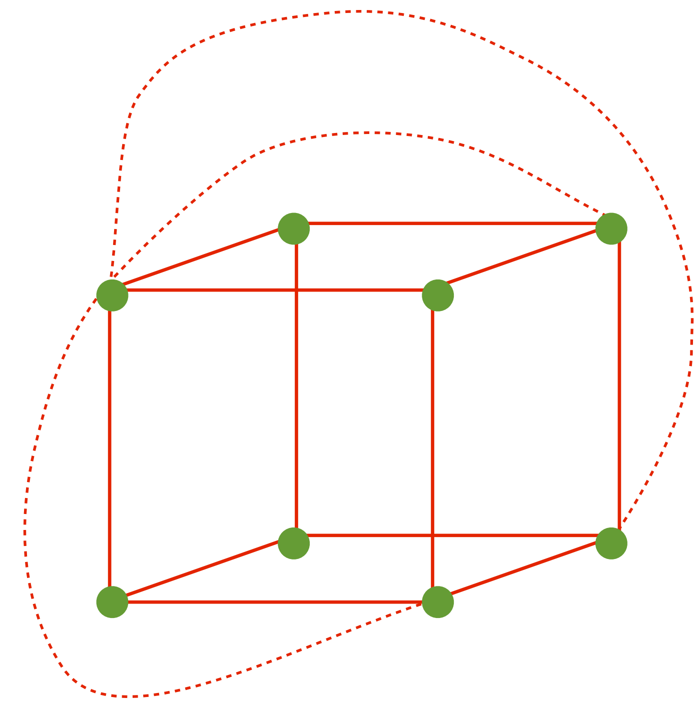
$$\Phi_G(S) = \epsilon$$



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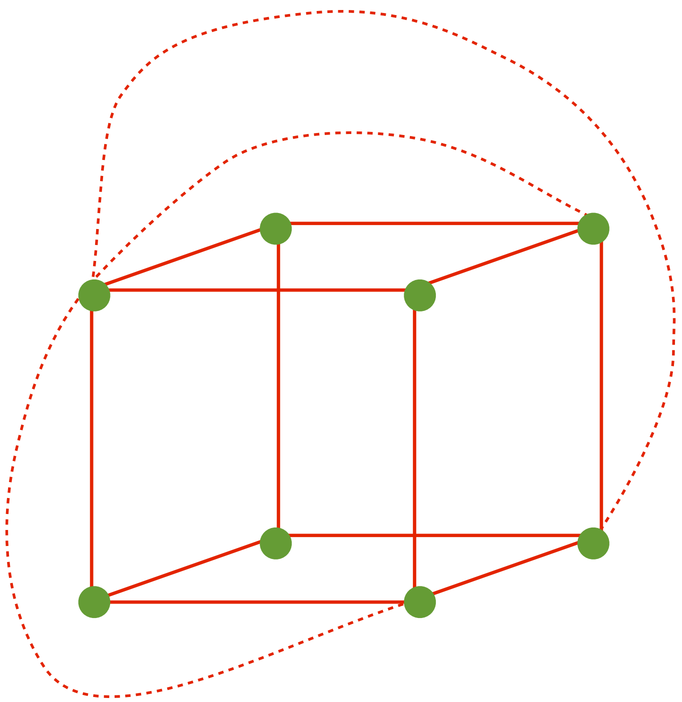
- Best cut with $\mu(S) = \delta$ is subcube of dimension $n - \log(1/\delta)$.

$$\Phi_G(S) \approx \epsilon \log(1/\delta)$$



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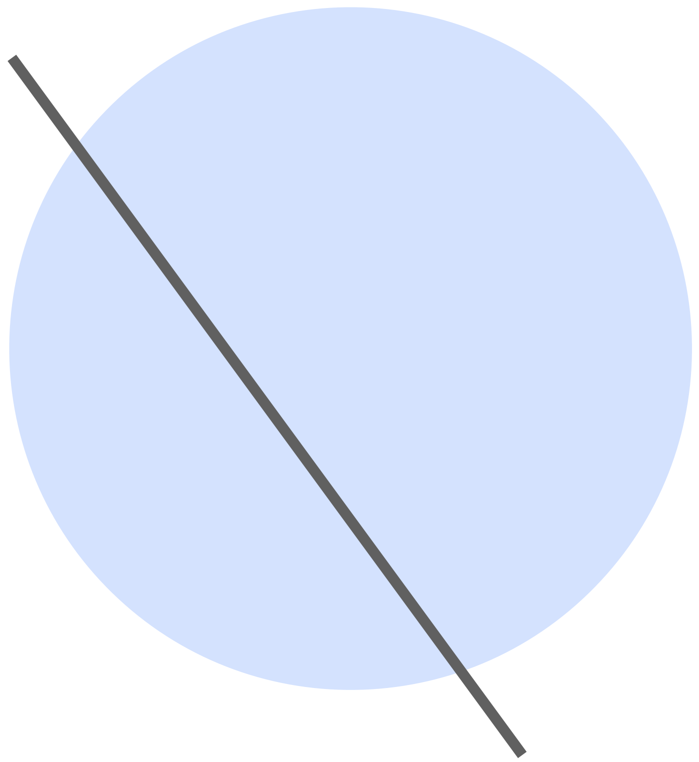

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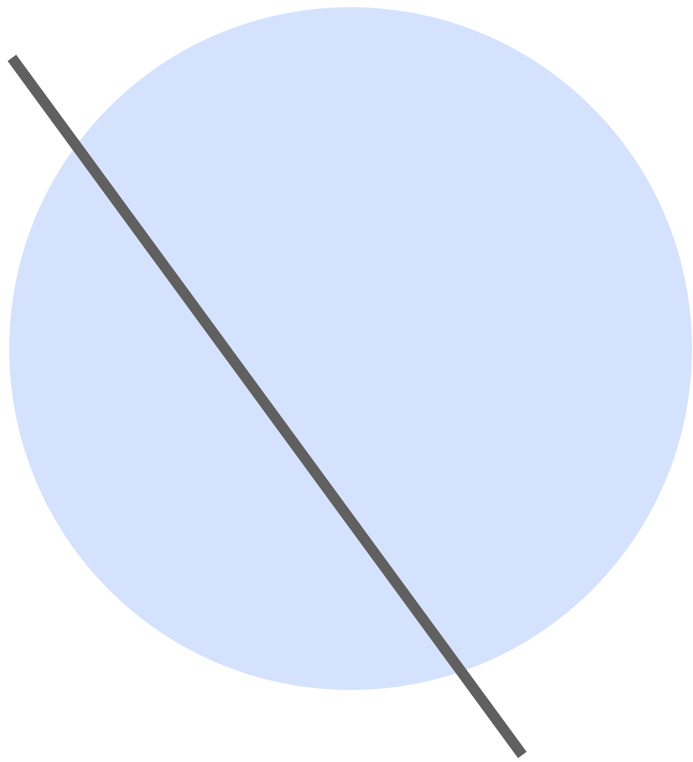

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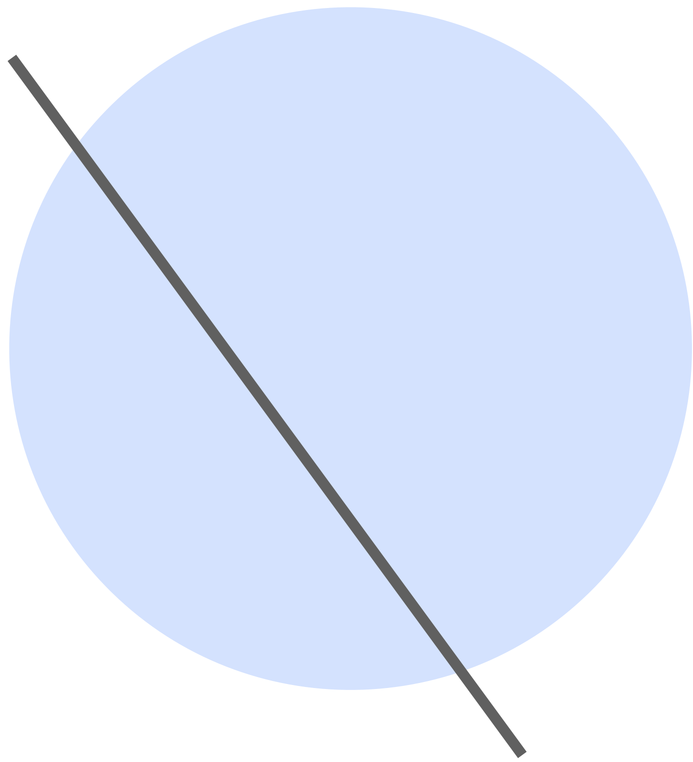

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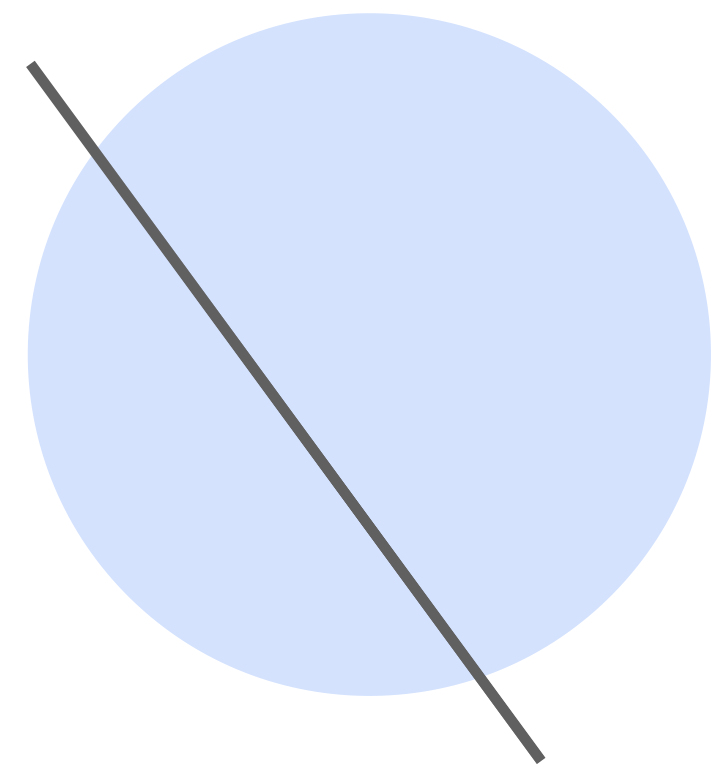
- Best cut with μ(S) = δ is threshold function with mean δ.

$$\Phi_G(S) \approx 1 - (1/\delta)\ \Gamma_{1-2\epsilon}(\delta) \approx 1 - \delta^{\epsilon/(1-\epsilon)}$$



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- SSE($\eta$,$\delta$): Given graph G, distinguish between the following cases:
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- SSE-Conjecture: $\forall \eta$ $\exists \delta$ such that SSE($\eta$,$\delta$) is hard.

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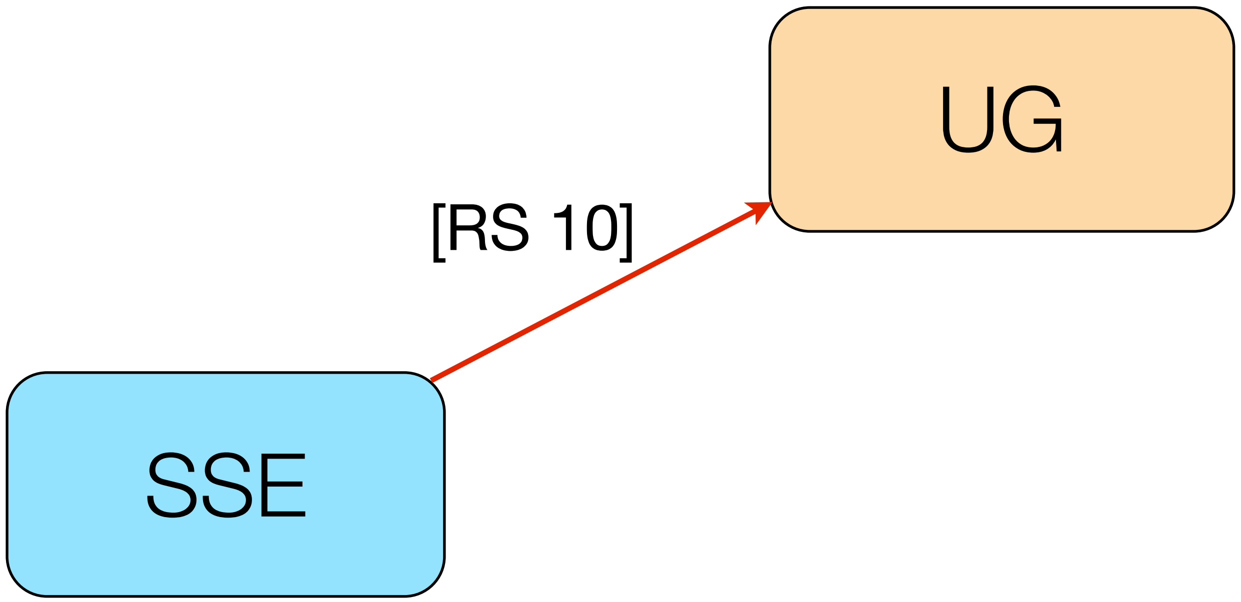

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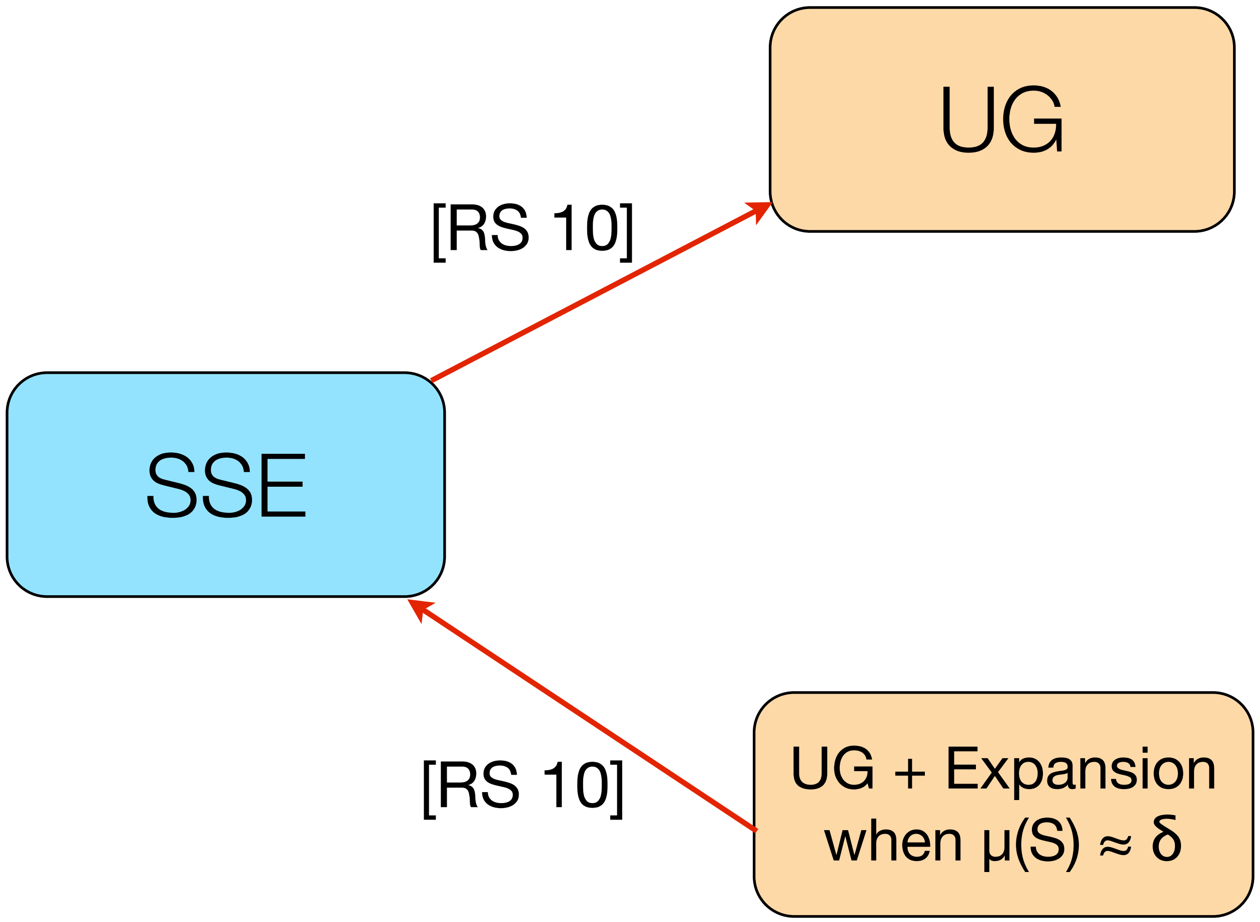

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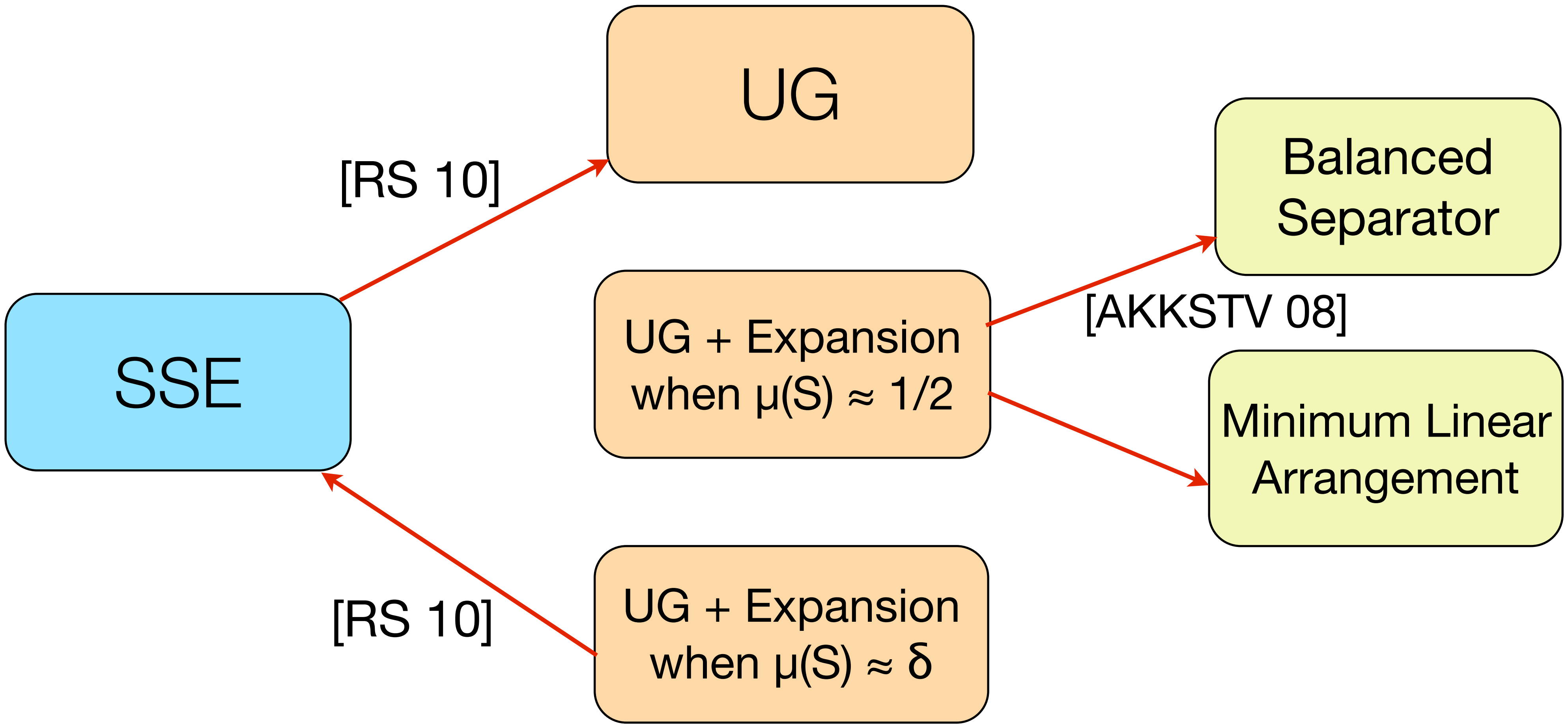

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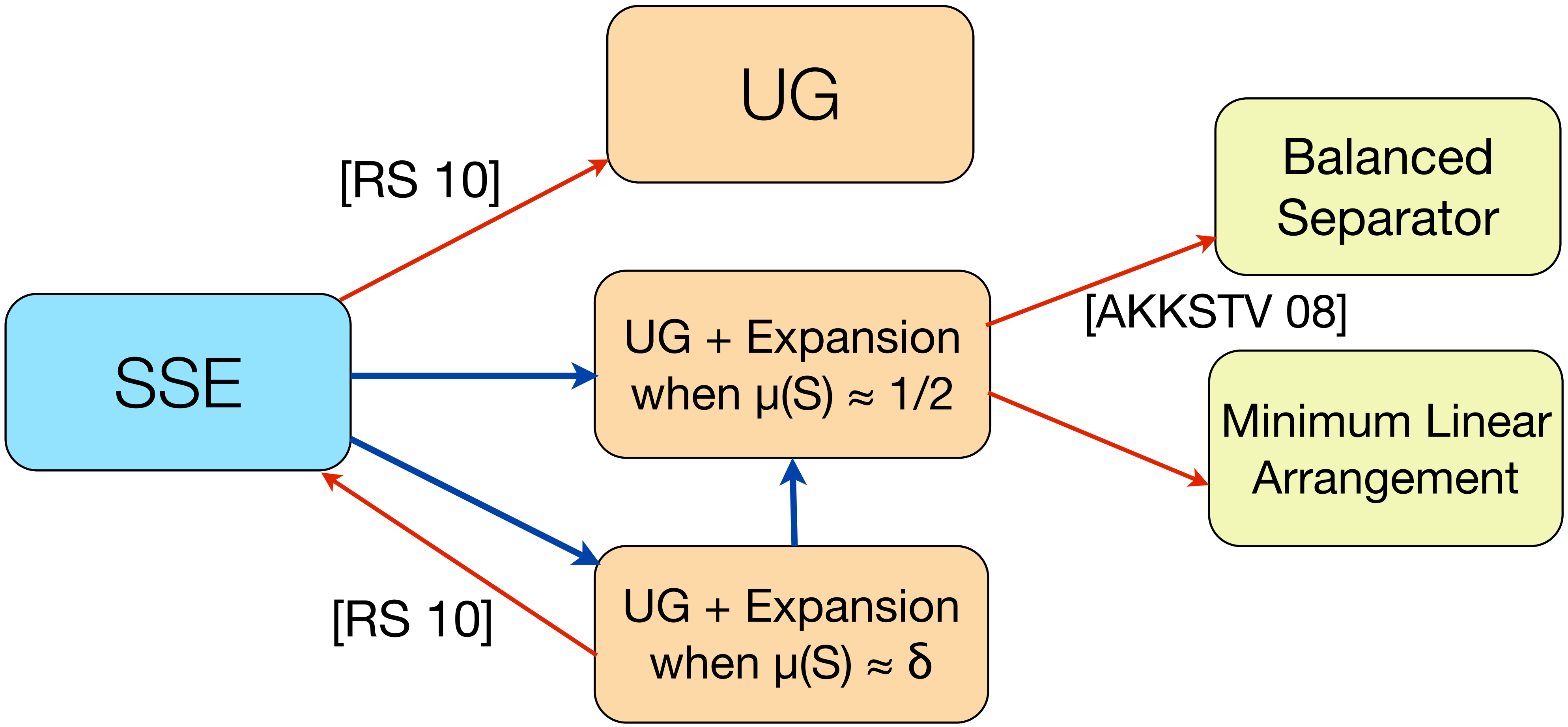

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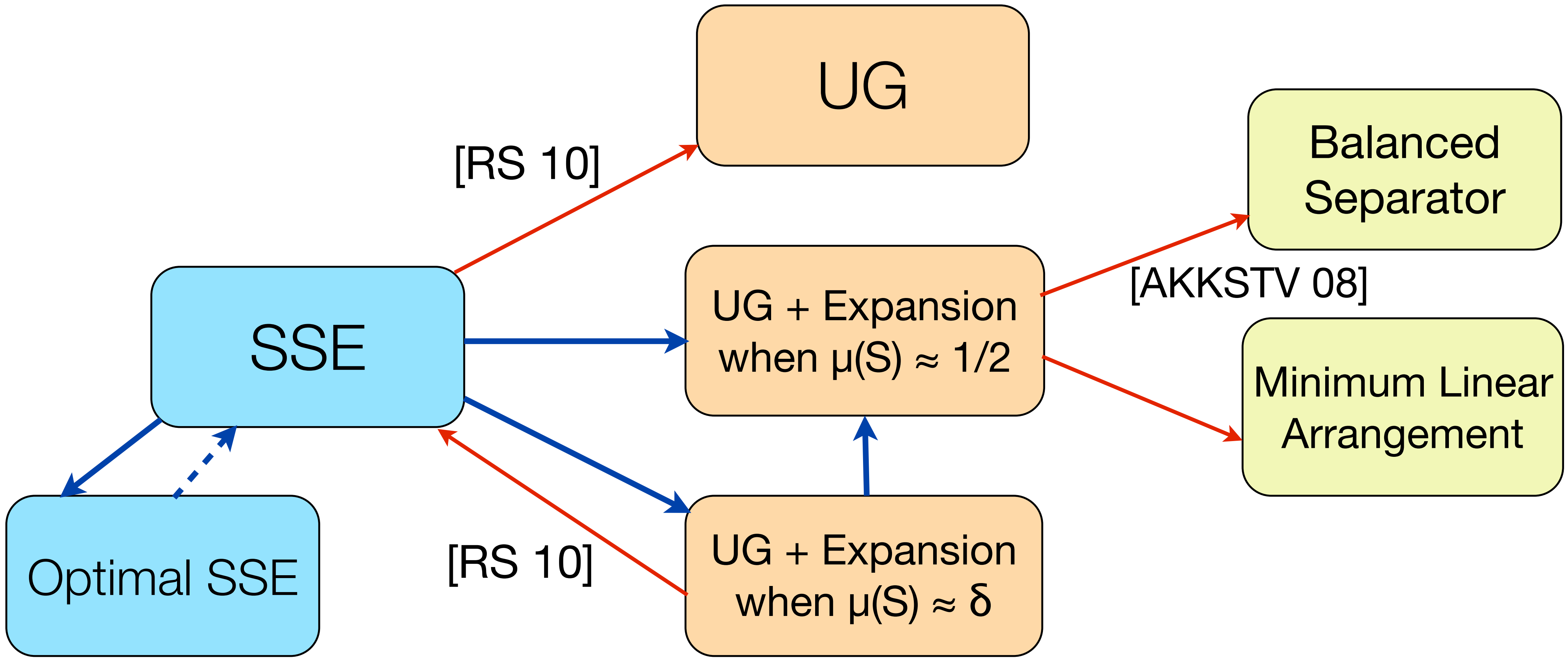

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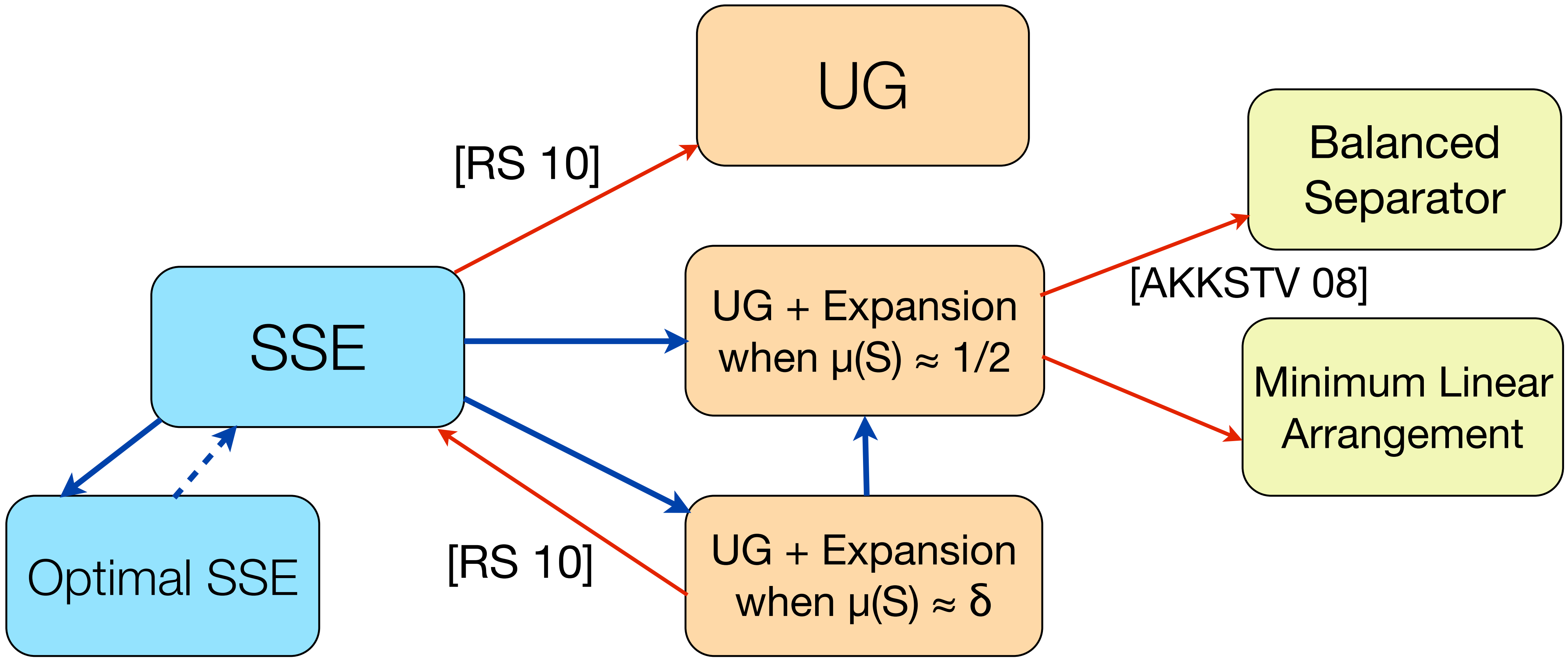


- Can reduce problem about expansion at scale $\delta$ to any scale $> \delta$.
- Similar to alphabet reduction for Unique Games by [KKMO 04] (UG alphabet $\approx 1/\delta$).

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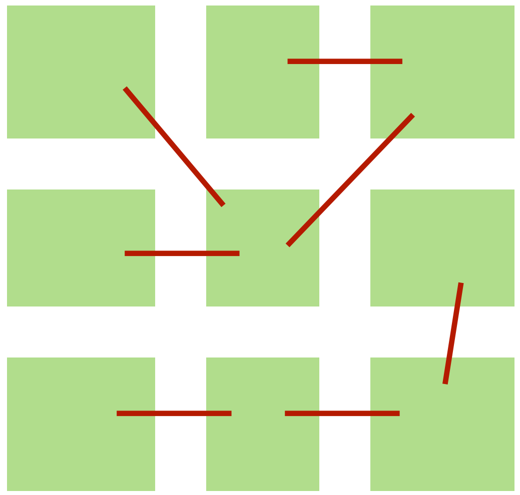

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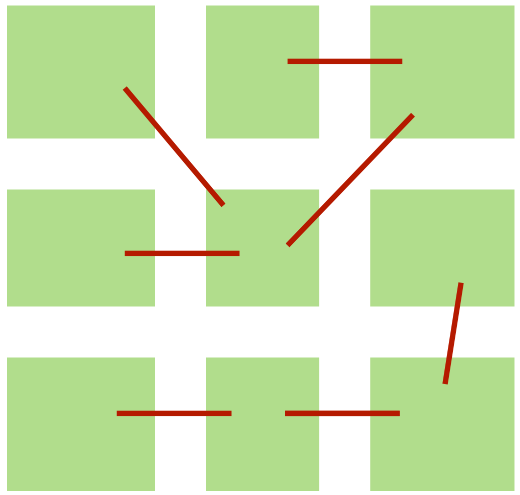

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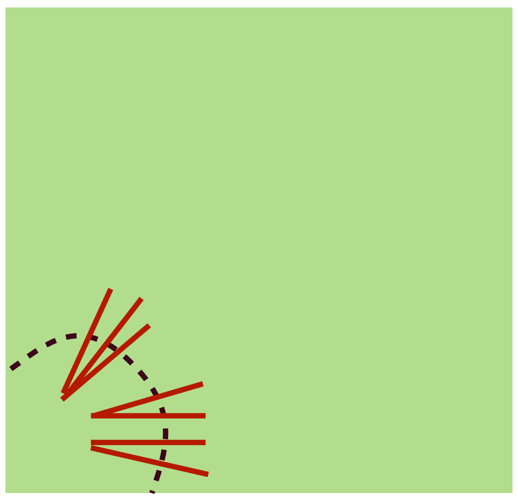

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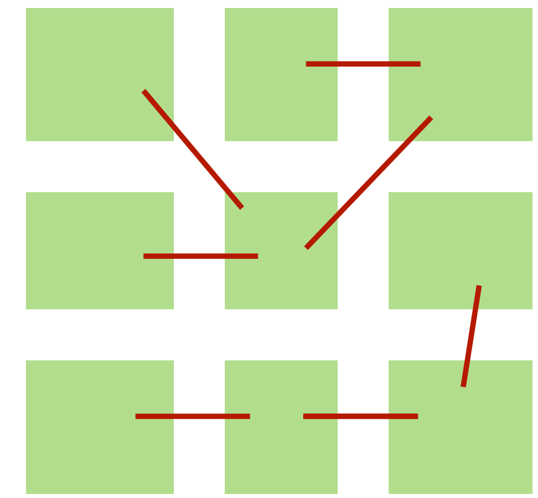

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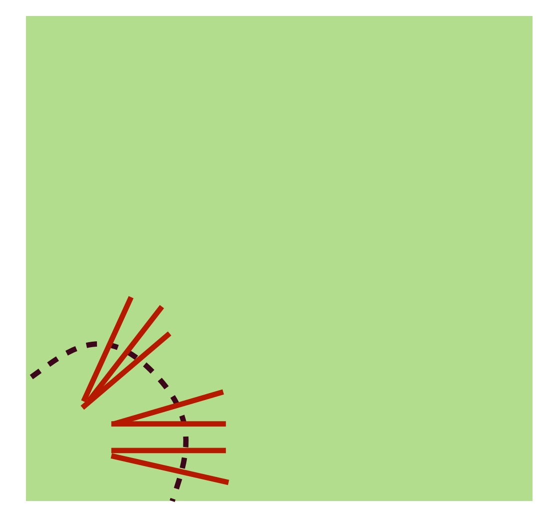

Matches (upto constants) the algorithmic guarantee of [RST 10]

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Also gives hardness for Minimum Bisection since $\mu(S) = 1/2$ in the first case.

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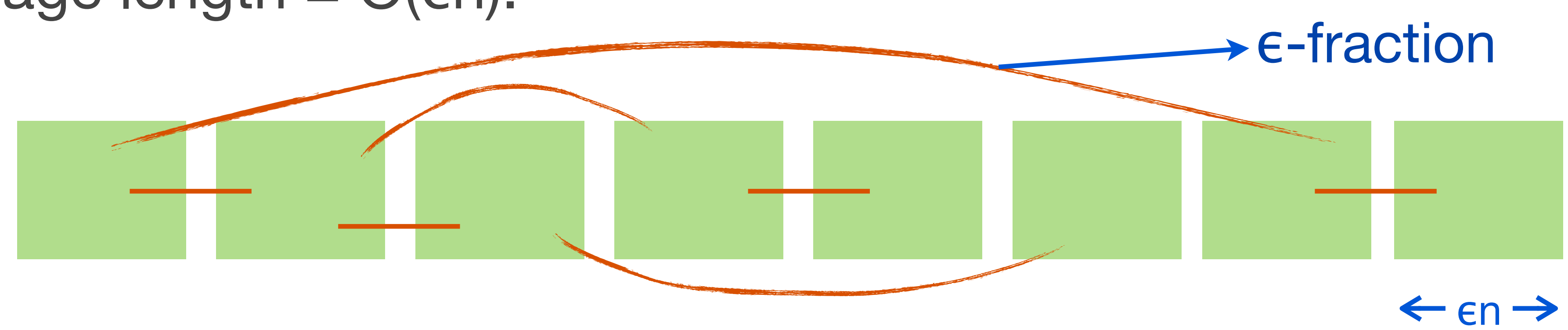

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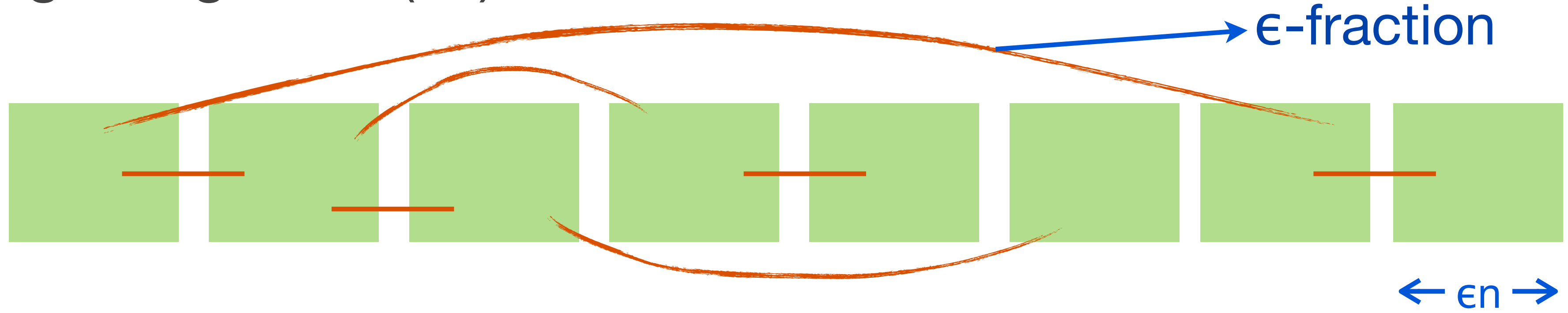
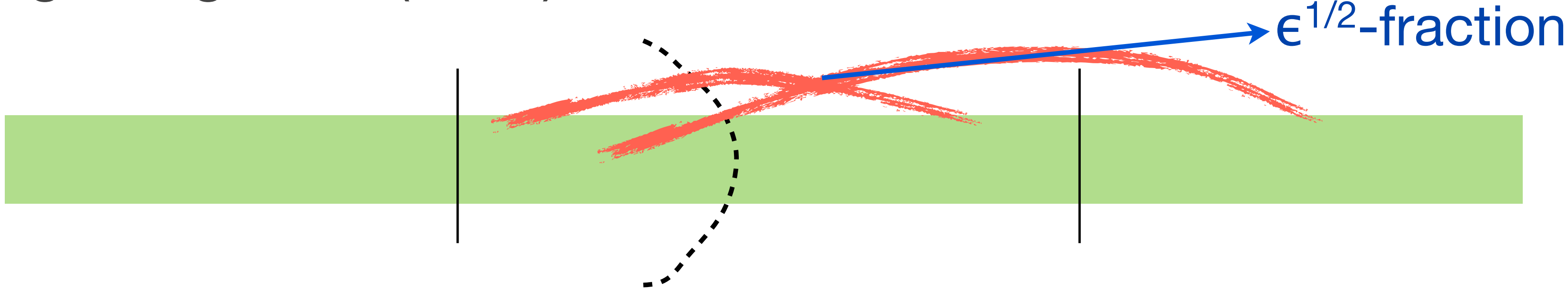



- $\Omega(\epsilon^{1/2})$ fraction of edges crossing every cut on middle n/3 vertices. Average length = $\Omega(\epsilon^{1/2}\, n)$.



# Reducing to an expansion problem

# Unique Games

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- Given: Graph G=(V,E), alphabet R and for each edge (u,v), a permutation constraint $\pi_{uv} : [R] \to [R]$.

  Find: A labeling $L : V \to [R]$ satisfying for as many edges (u,v) as possible

  $\pi_{uv}(L(u)) = L(v)$

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- UGC: Hard to distnguish between the cases when
  - At least 1-η fraction of edges can be satisfied
  - At most η fraction of edges can be satisfied.

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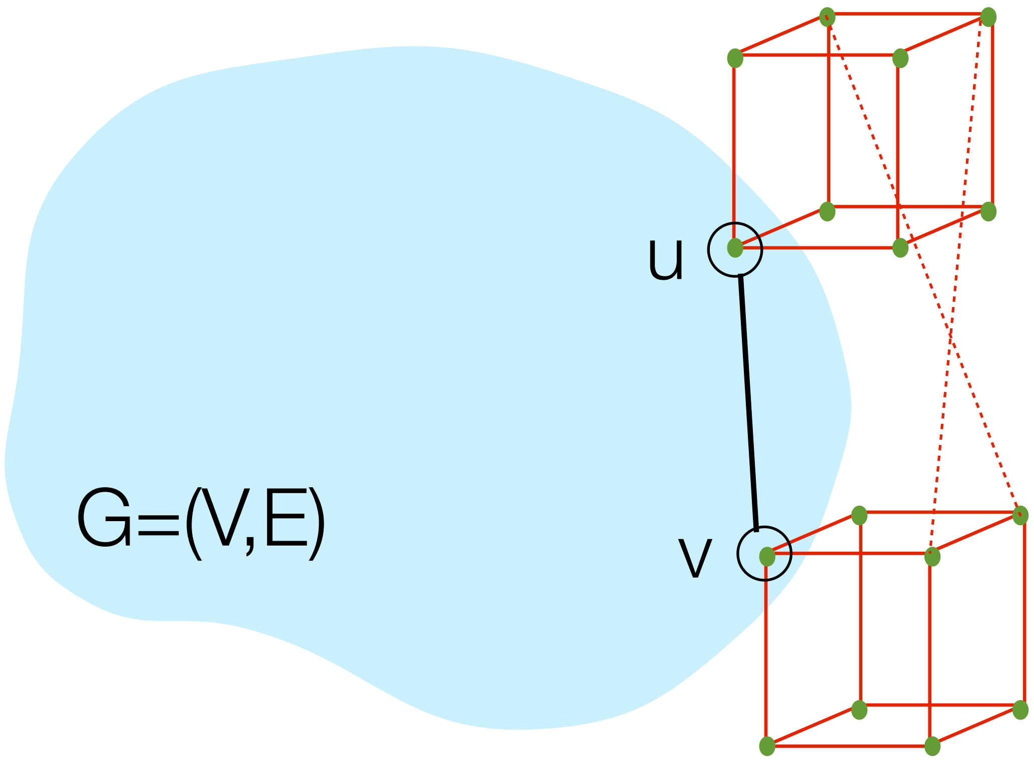


- Attach $\{0,1\}^R$ to each vertex of G
- Pick $x, y \in \{0,1\}^R$ with $\epsilon$-noise (weight = $\epsilon^{H(x,y)} (1-\epsilon)^{R-H(x,y)}$ )
- Connect $(u, \pi_{uv}(x))$ to $(v,y)$ with above weight. (send $(u, \pi_{uv}(x))$ to Alice, $(v,y)$ to Bob)
- Ask for a balanced cut on new graph.

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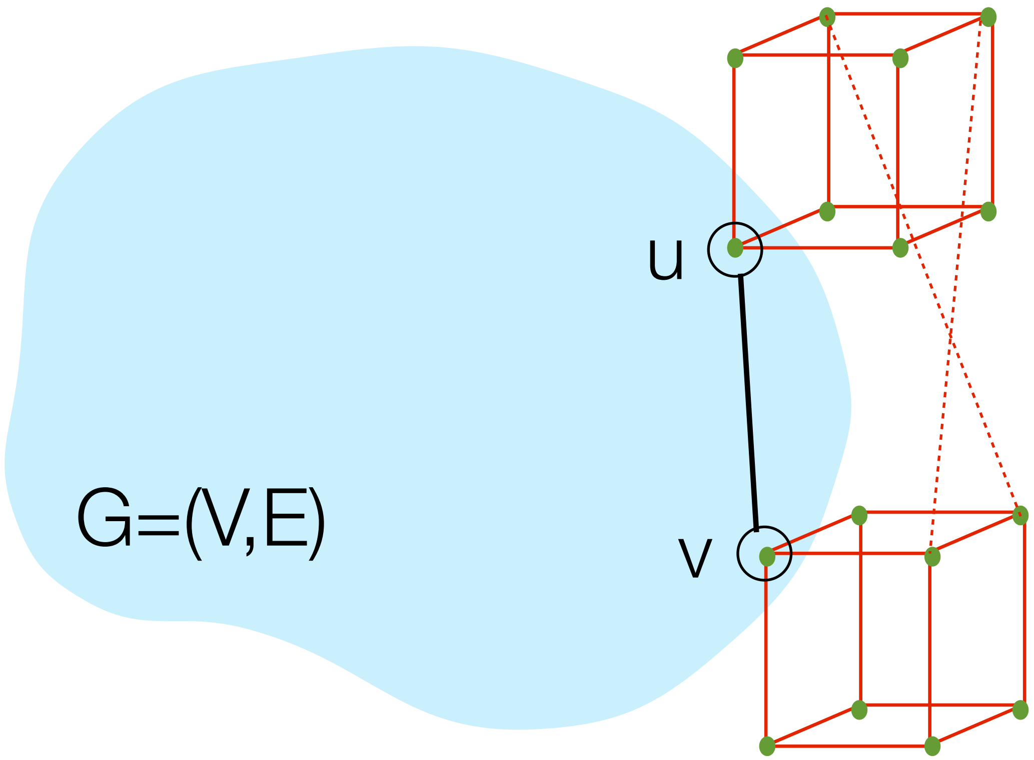


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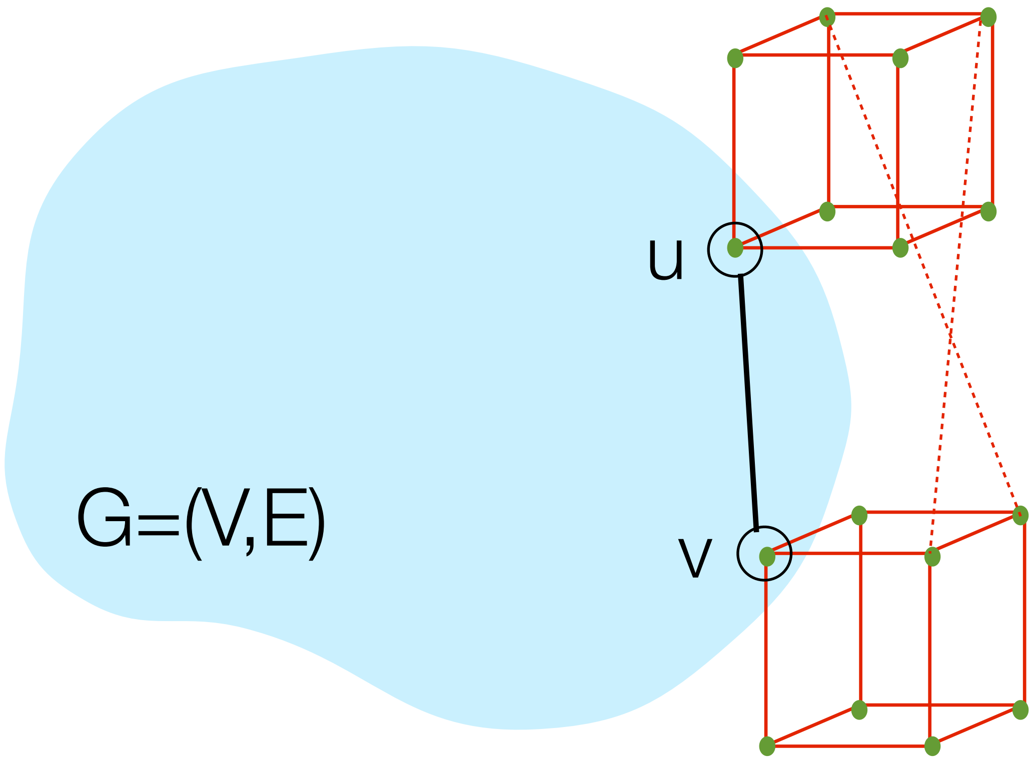


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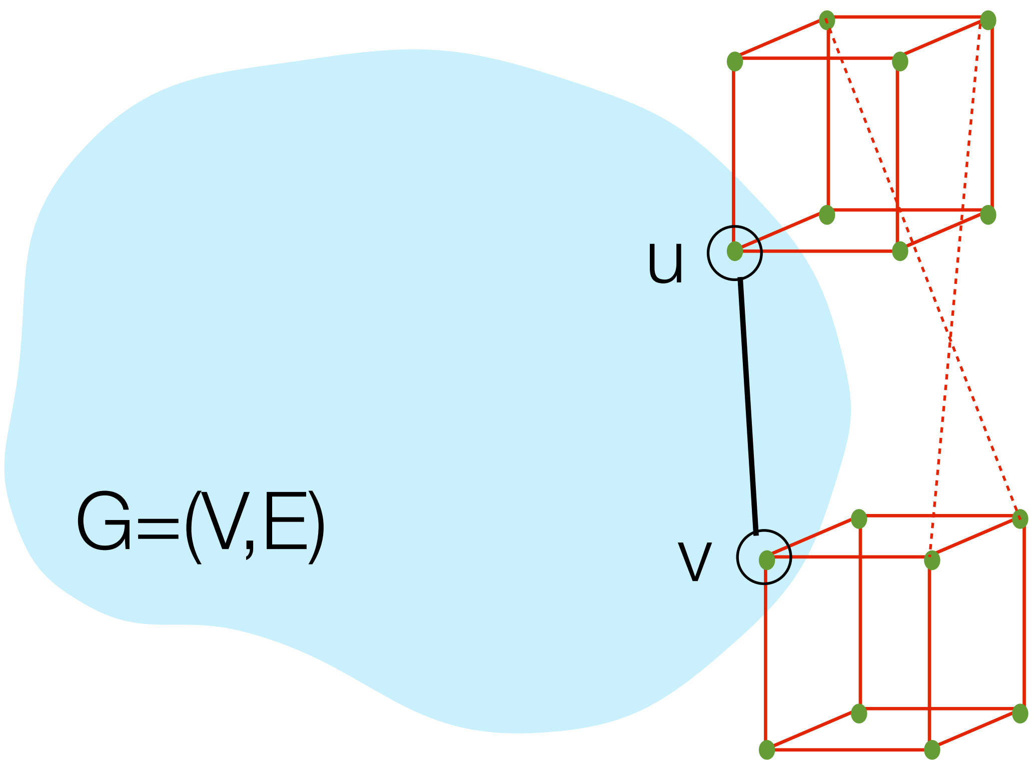
- If $\pi_{uv}(L(u)) = L(v)$, then only $\epsilon$-fraction of edges between cubes of u and v are cut.
- If labeling satisfies $1-\eta$ fraction, then value of cut in graph is $\leq \epsilon + \eta$.

# Problems with the reduction



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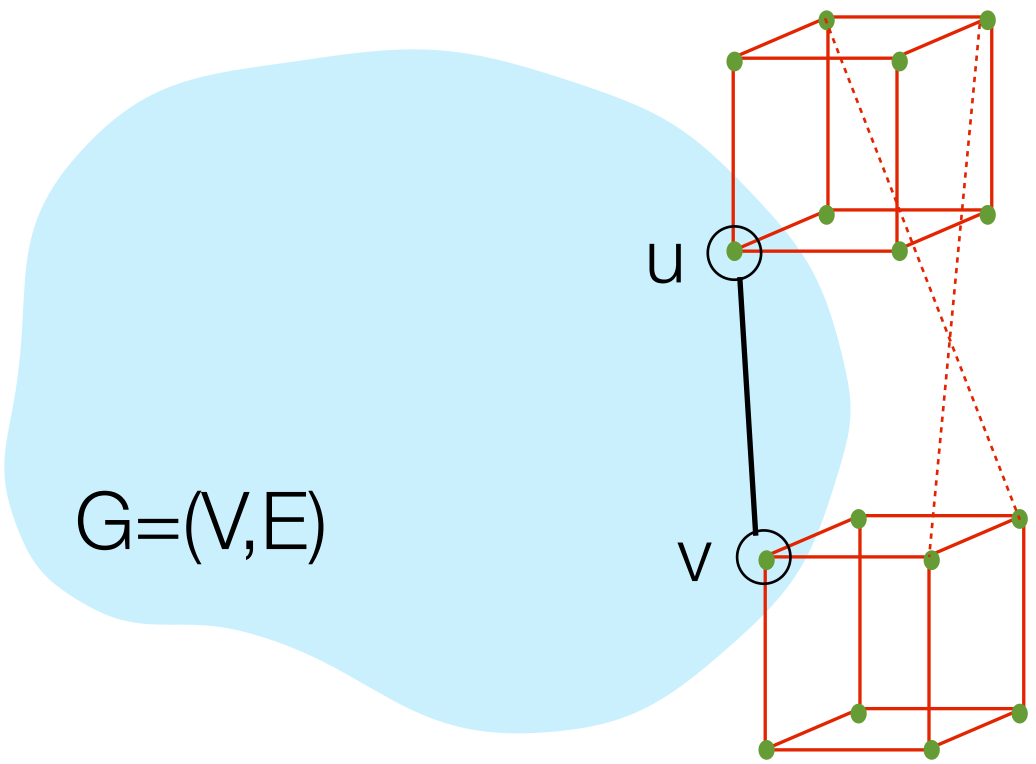


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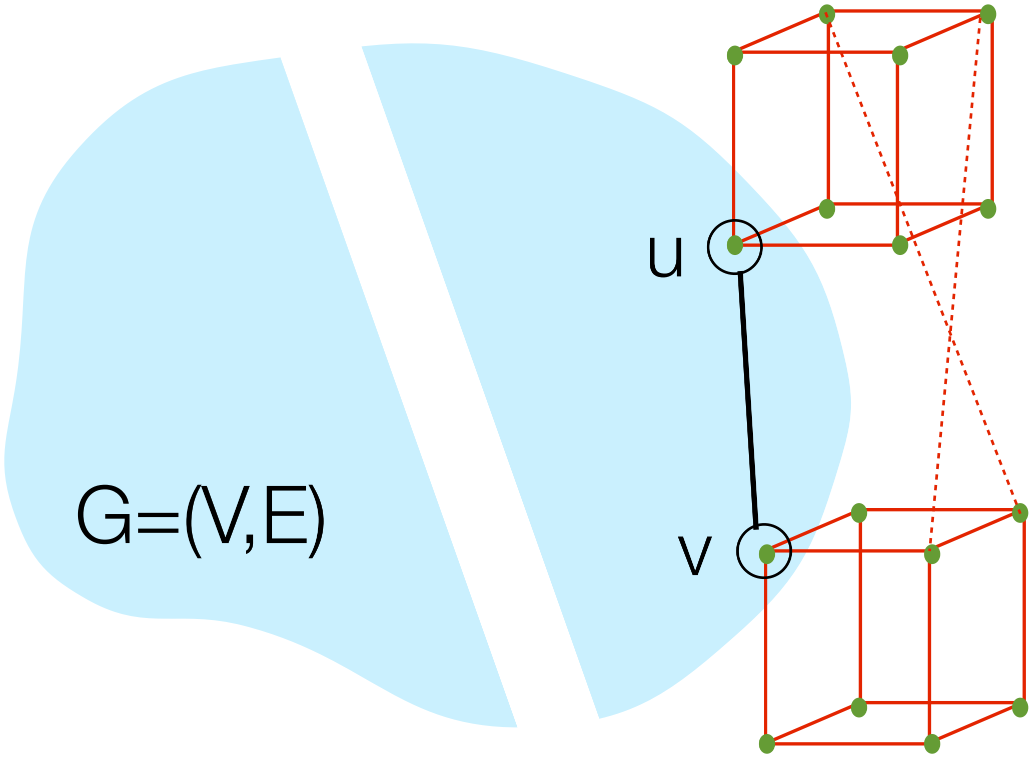


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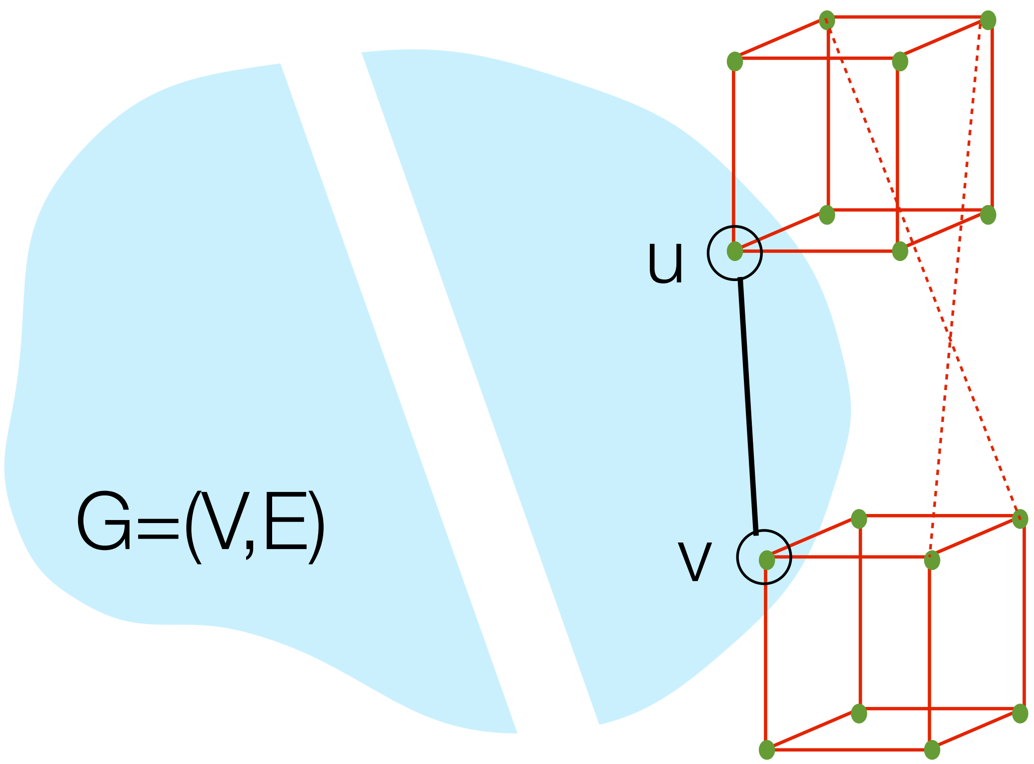


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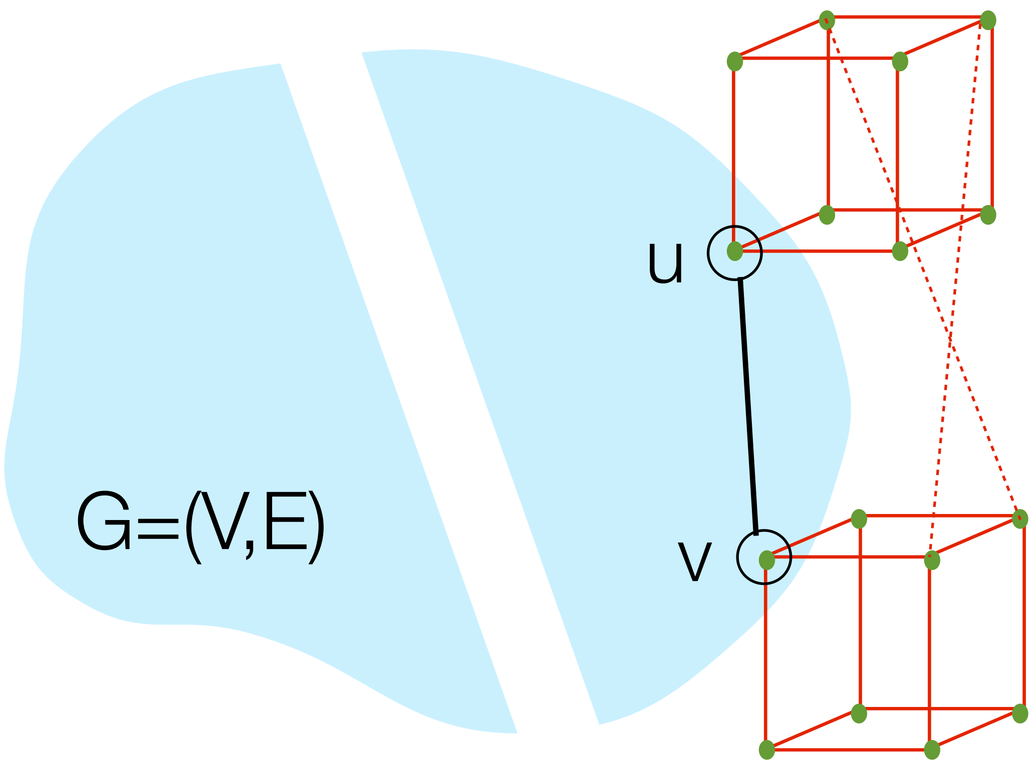


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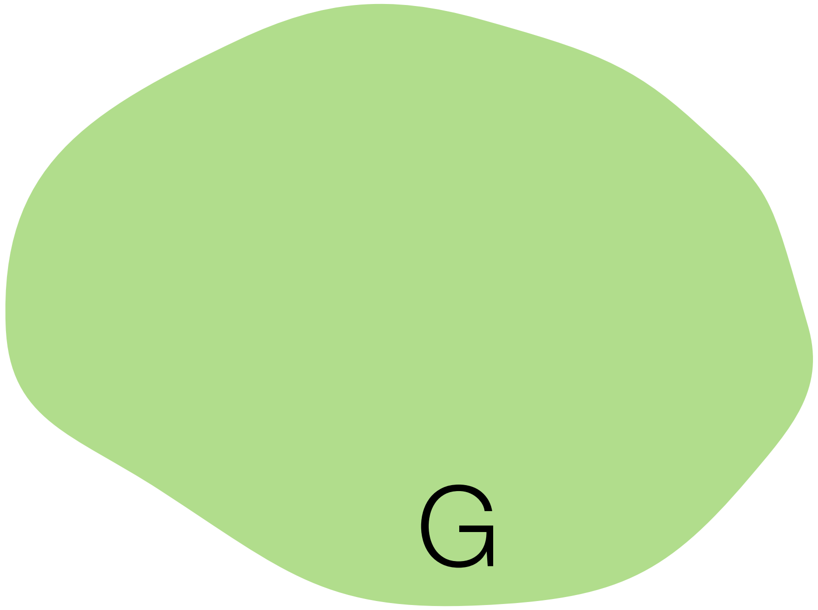
- For UGs obtained by reducing from Small-Set Expansion, can introduce a special folding operation, which does not preserve structure of UG-graph.
- Even though the graph for UG may not have expansion, the graph for Balanced Separator does.

# From SSE to Unique Games [RS 10]



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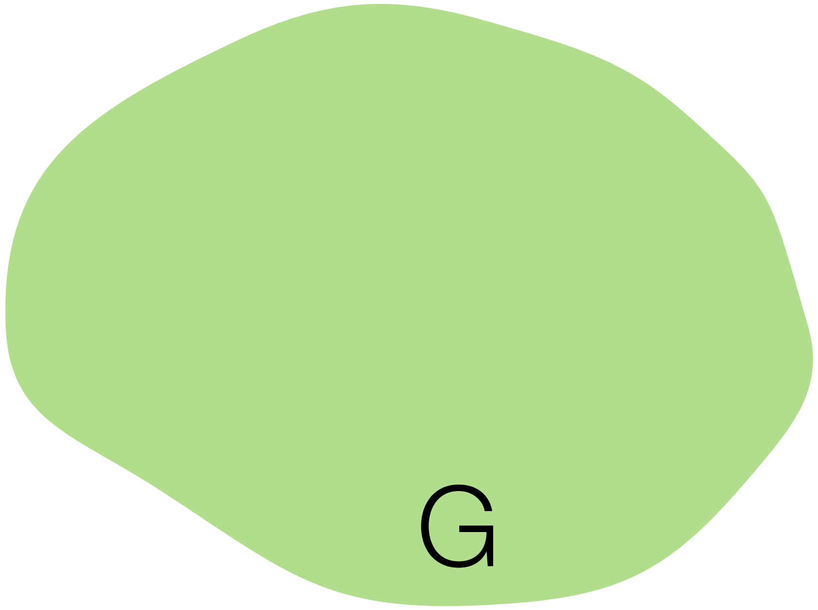


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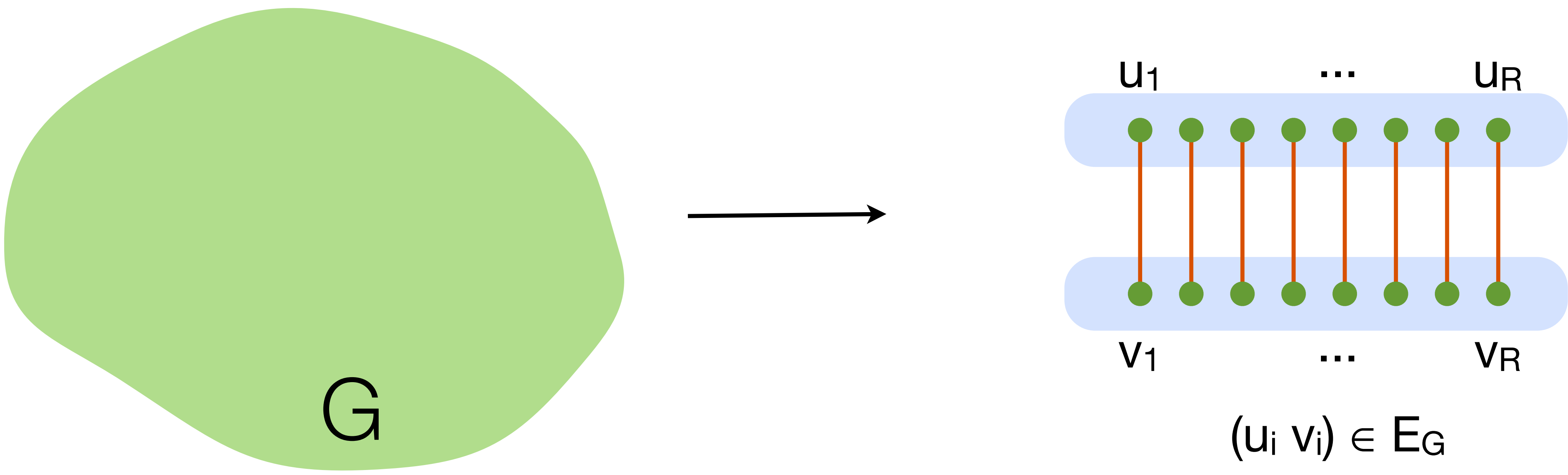


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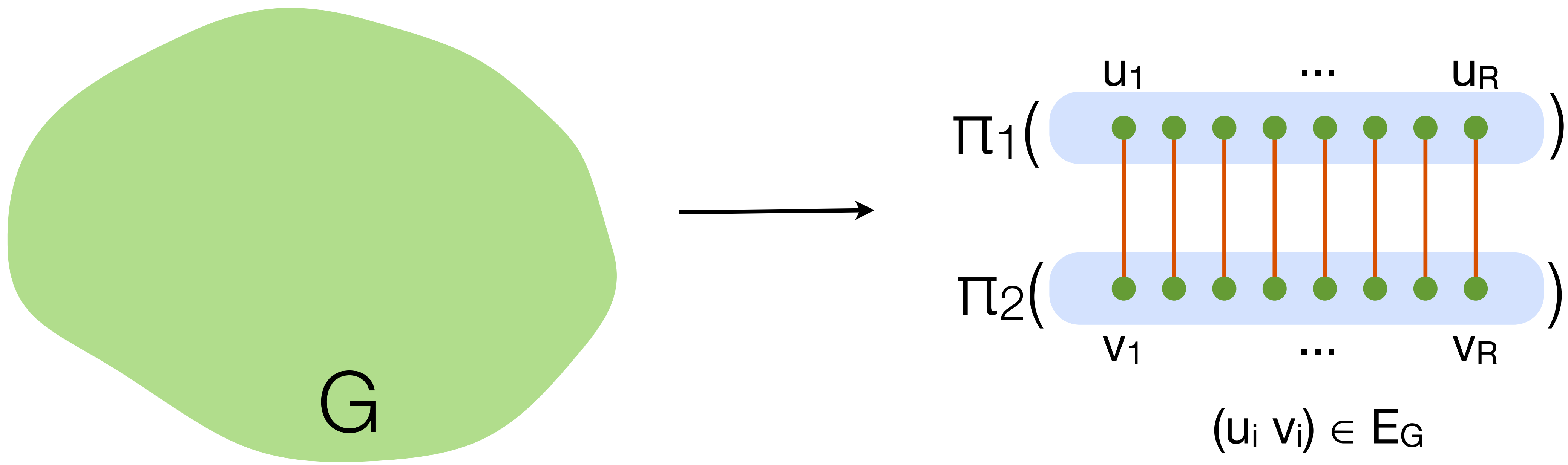
- Pick $(A,B) \sim E_G{}^R$

# From SSE to Unique Games [RS 10]




- UG is defined on $G^R$ and has alphabet size R.
- Pick (A,B) ~ $E_G^R$
- Pick random permutations $\pi_1$, $\pi_2$ from [R] to [R]
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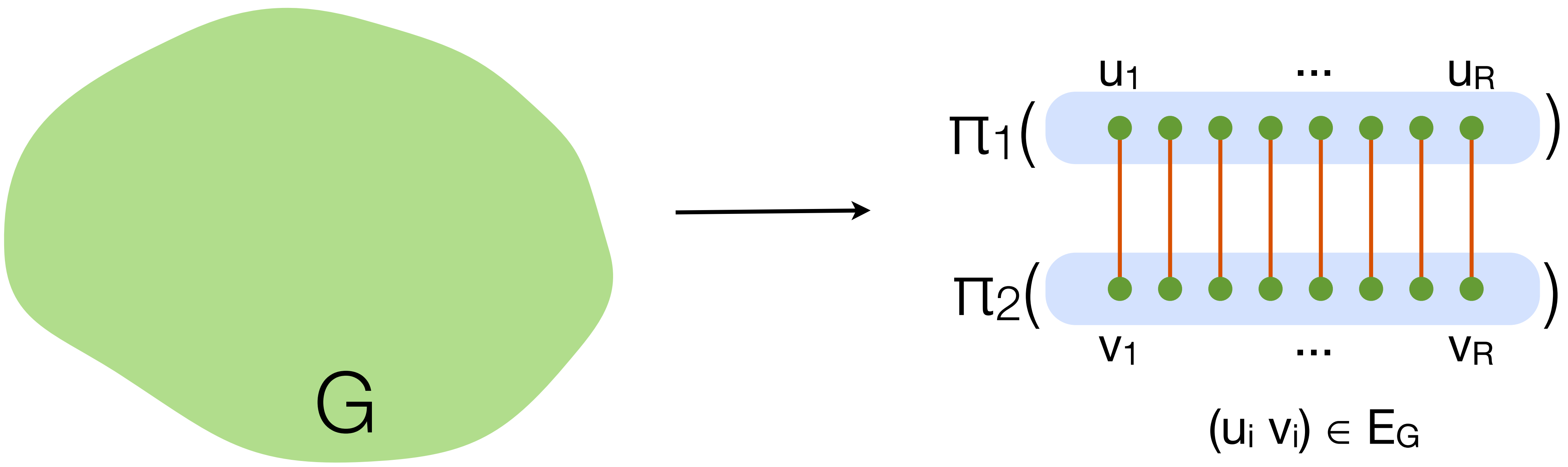


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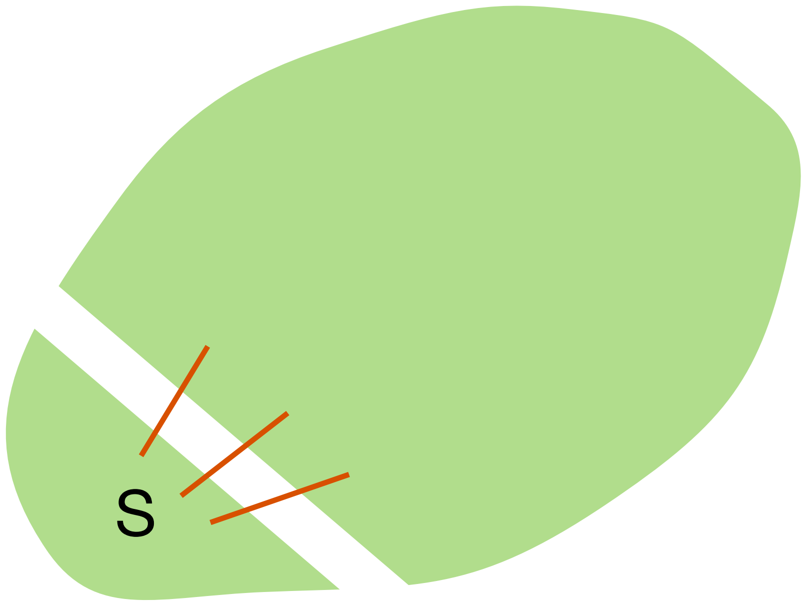
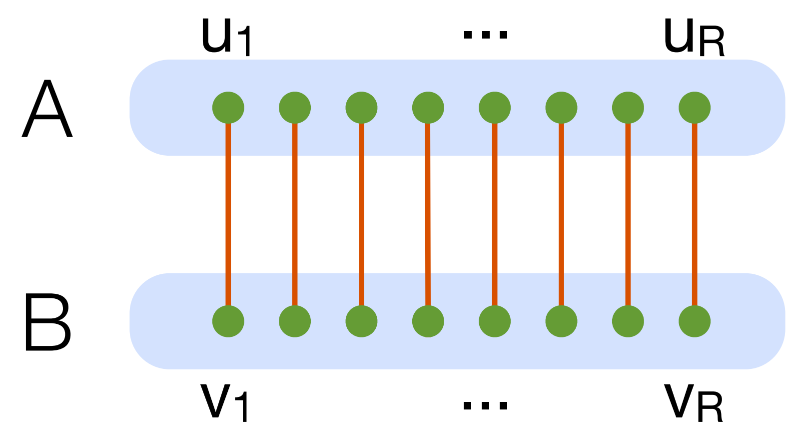
- Connect $\pi_1$(A) to $\pi_2$(B). Call them u,v. (send $\pi_1$(A) to Alice, $\pi_2$(B) to Bob)
- Labeling is required to find an edge in G. Constraint on (u,v) is

$$\pi_1^{-1}(L(u)) = \pi_2^{-1}(L(v))$$

# Why the SSE reduction works




$\mu(S) = 1/R, \;\; \Phi_G(S) \leq \eta$



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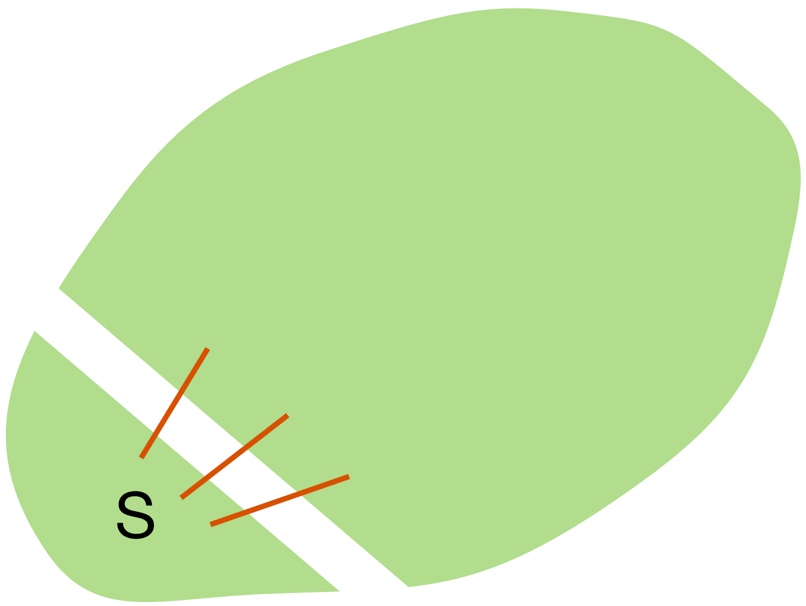


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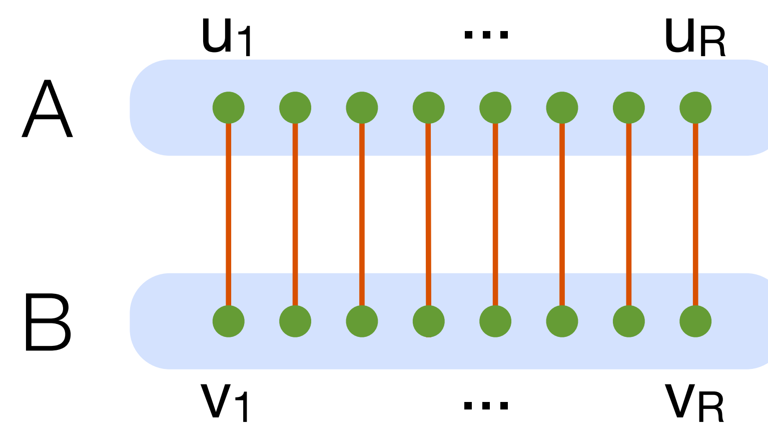


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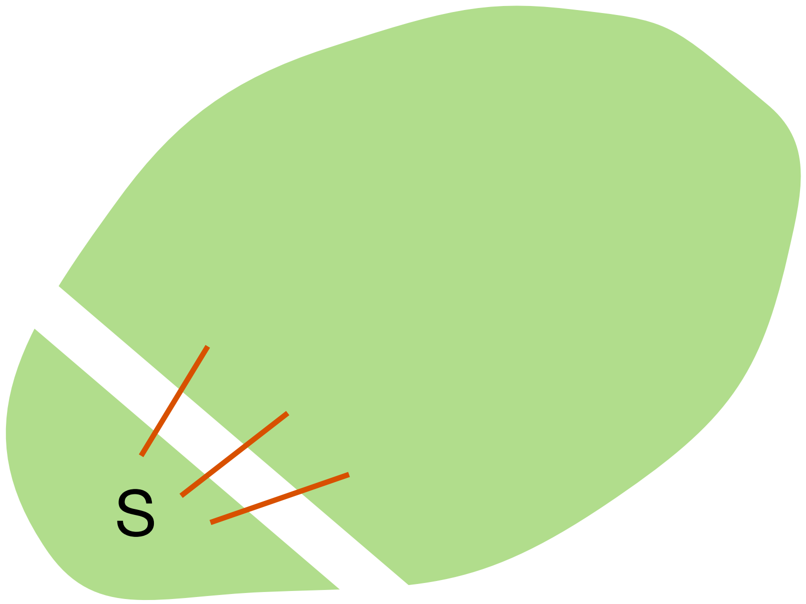


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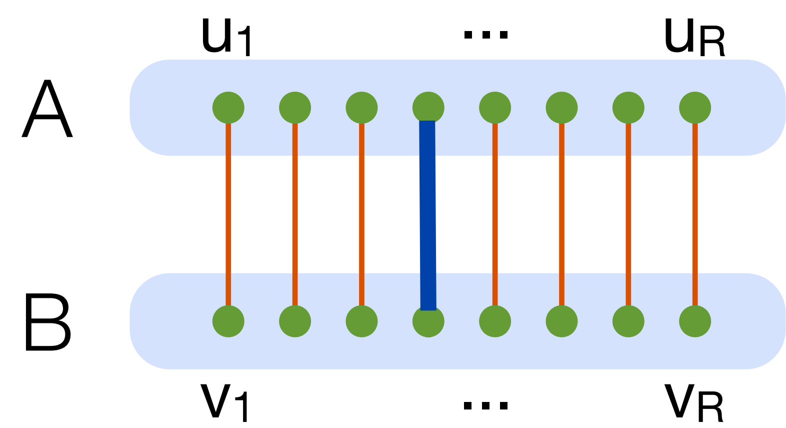


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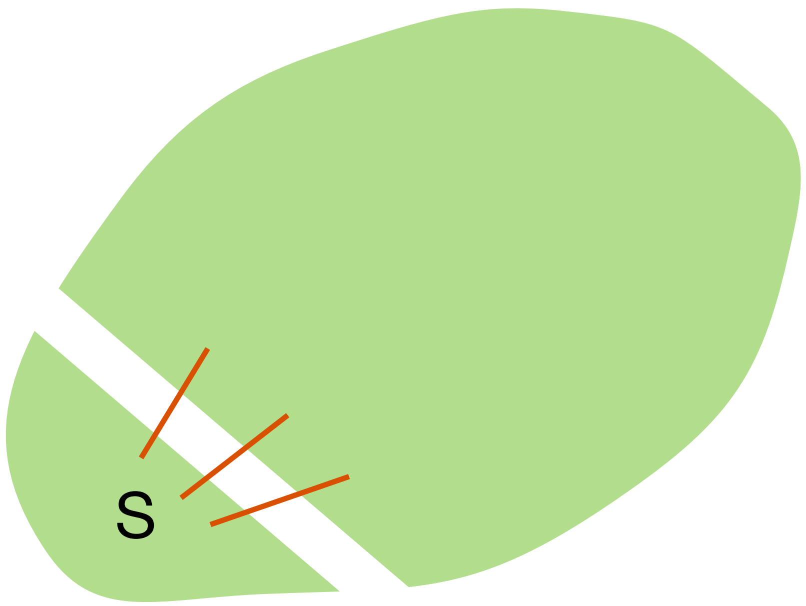


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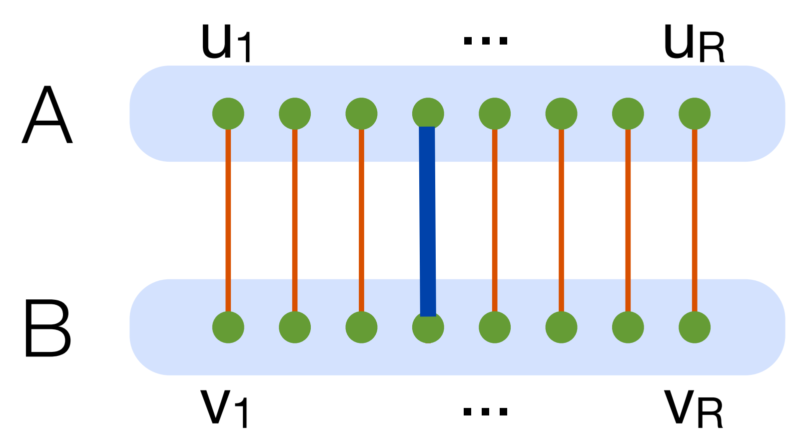


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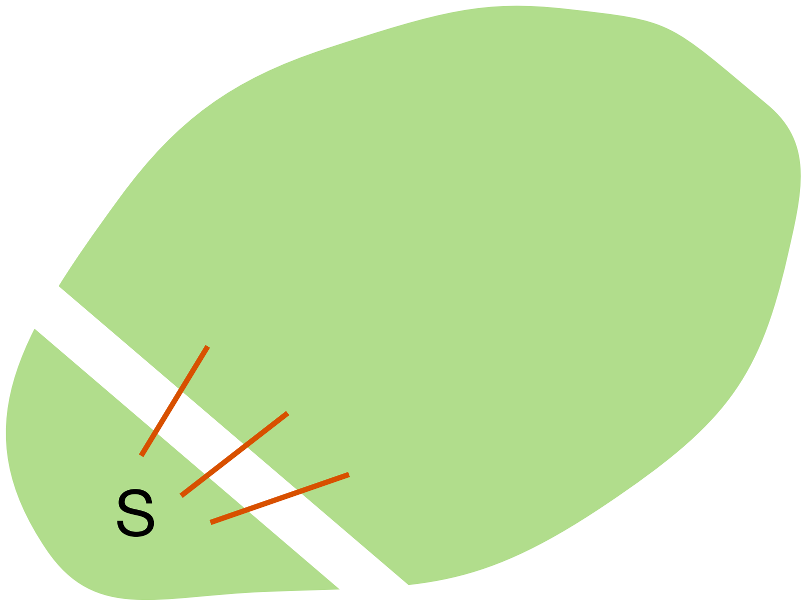


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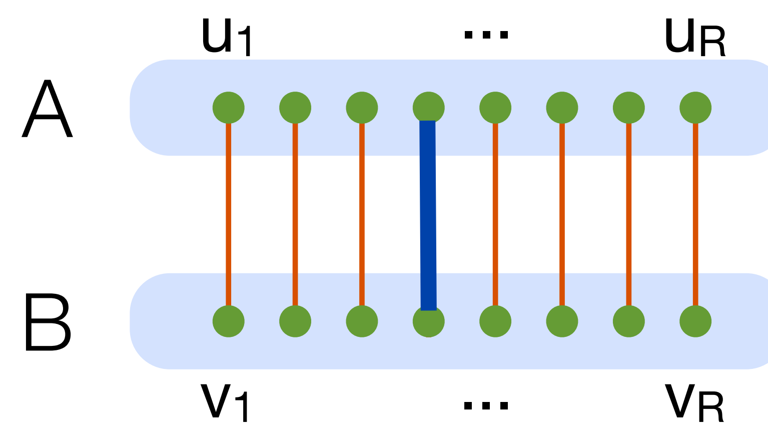


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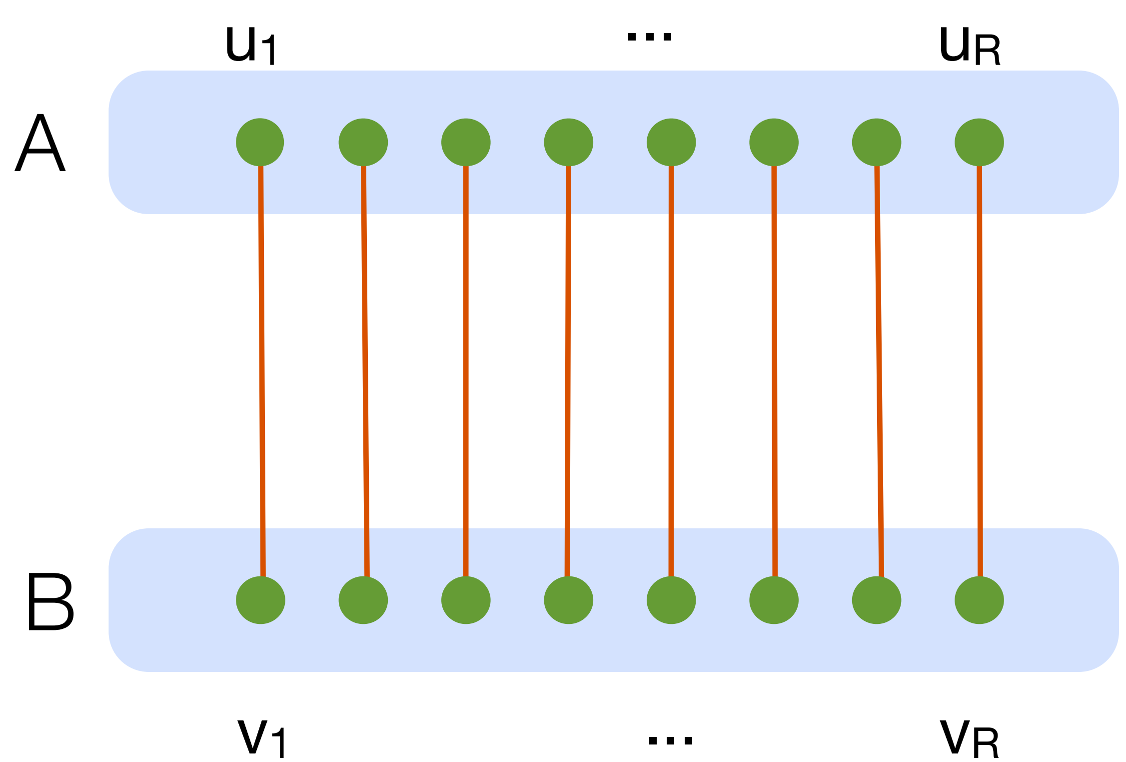
- Works with probability $(1-\eta)/e$

# Combining the two reductions



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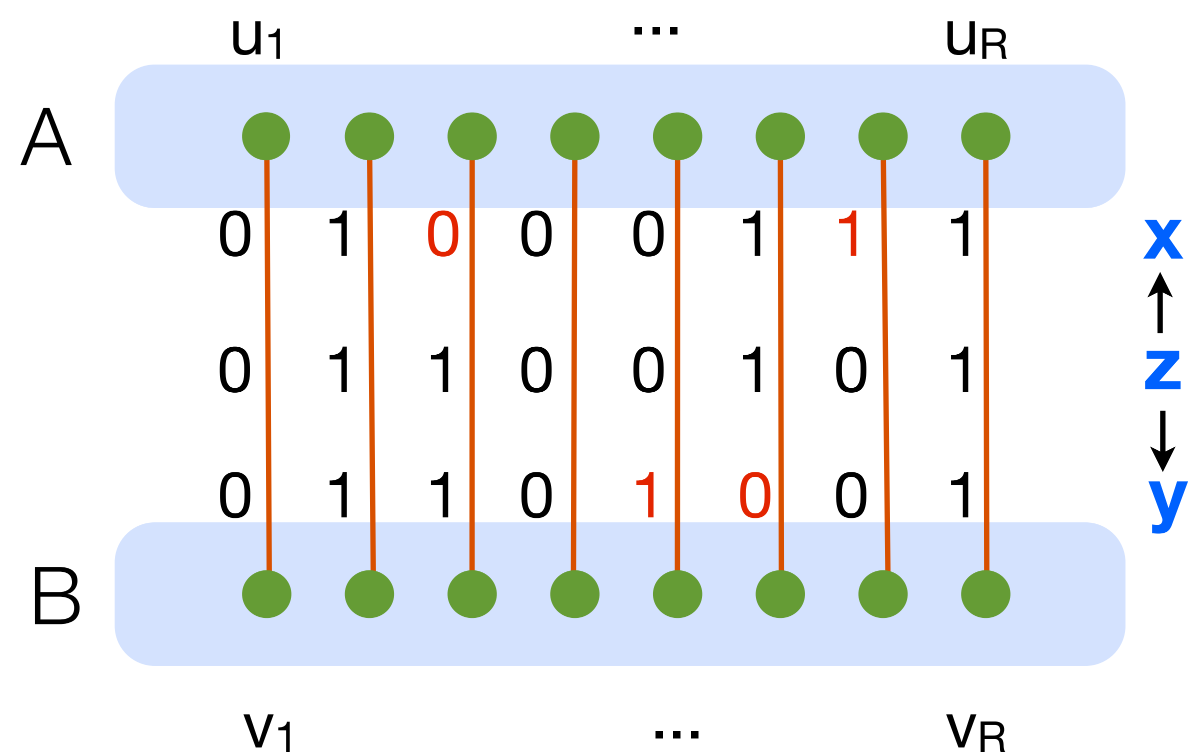

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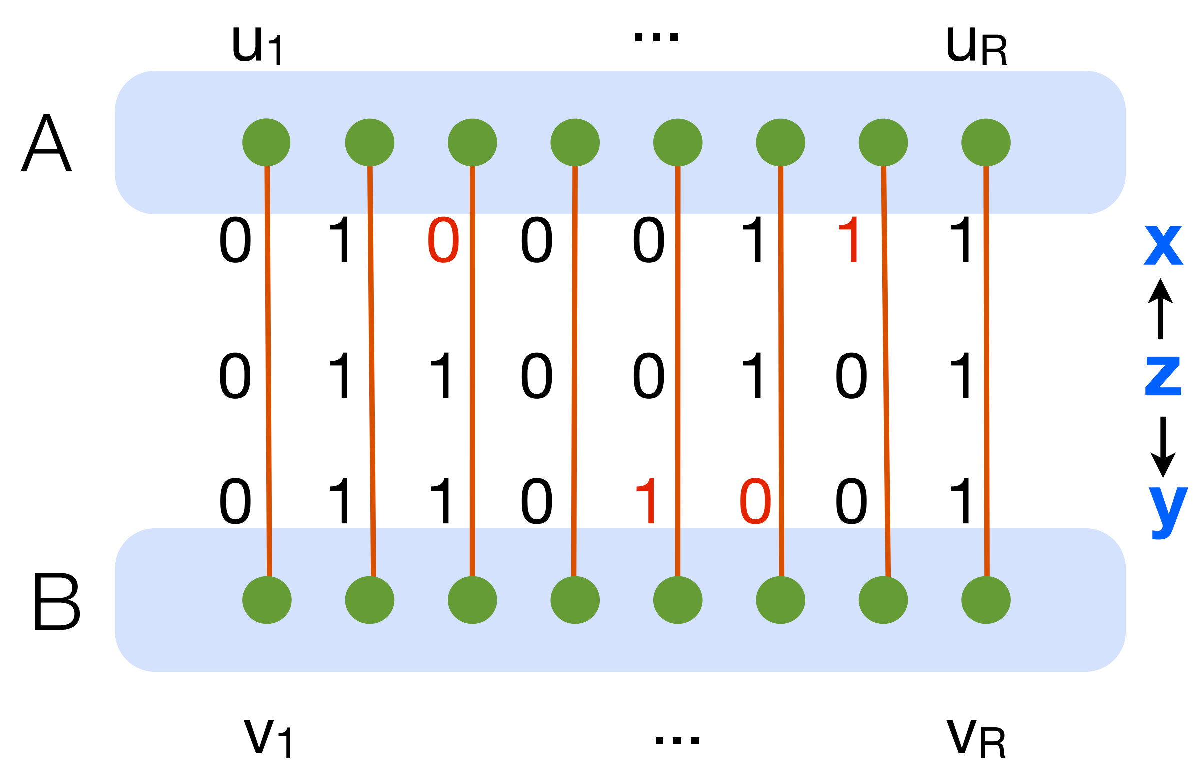


Pick random $z \in \{0,1\}^R$

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Connect $(\pi_1(A), \pi_1(x))$ to $(\pi_2(B), \pi_2(y))$.

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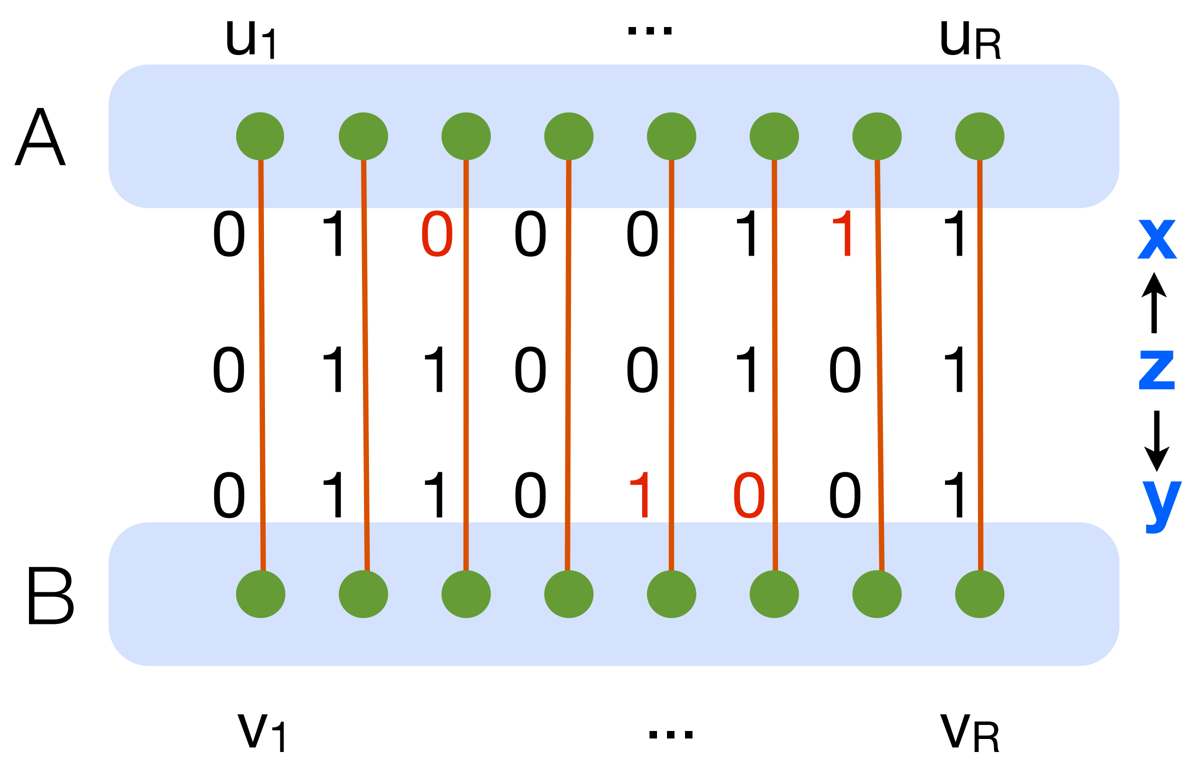


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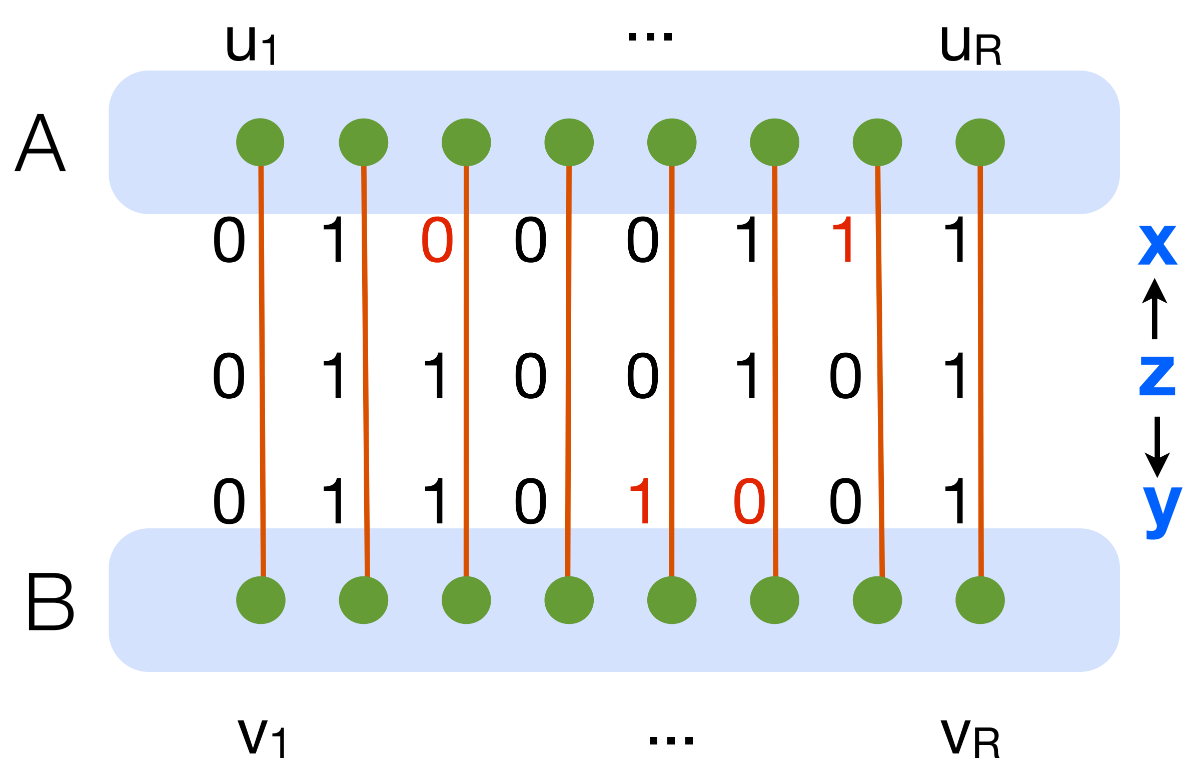


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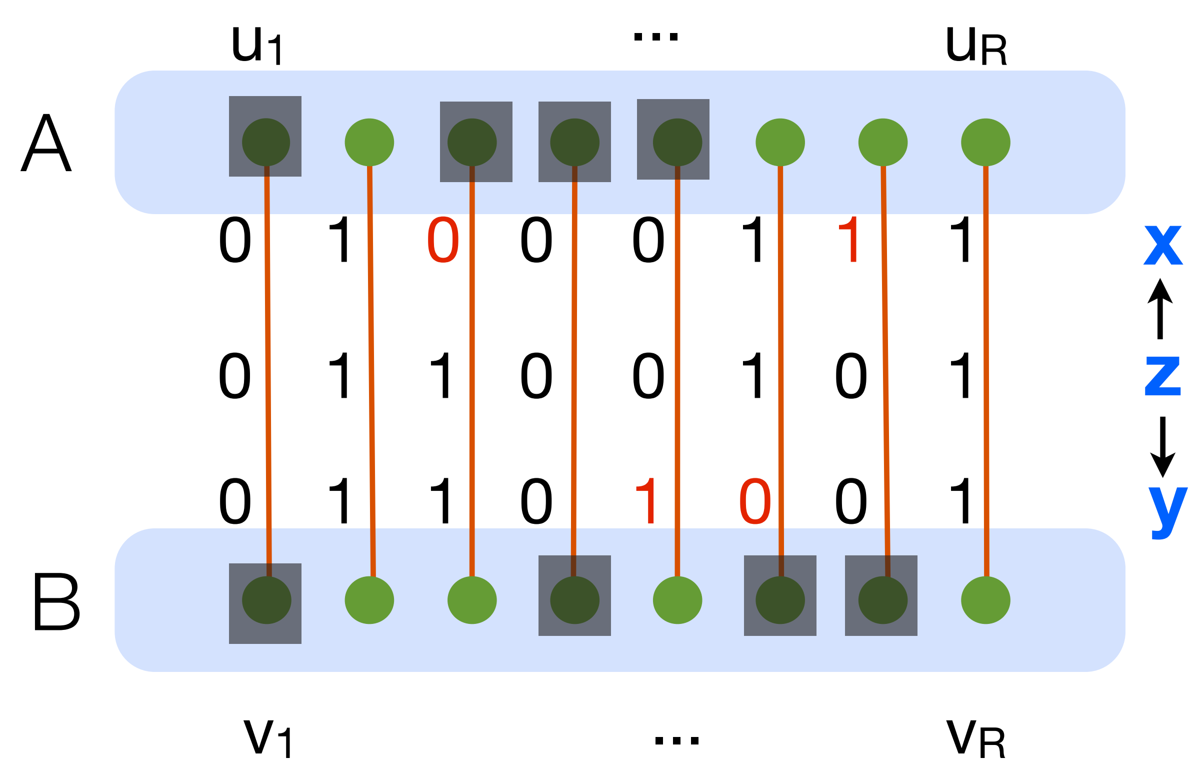


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- Think of bits as picking out subsets of A and B.
- Modification: Only send these subsets. Hide elements of A (or B) in positions where x (resp. y) are 0 (send $\perp$ instead).
- Corresponds to a folding operation on the graph obtained after reduction.

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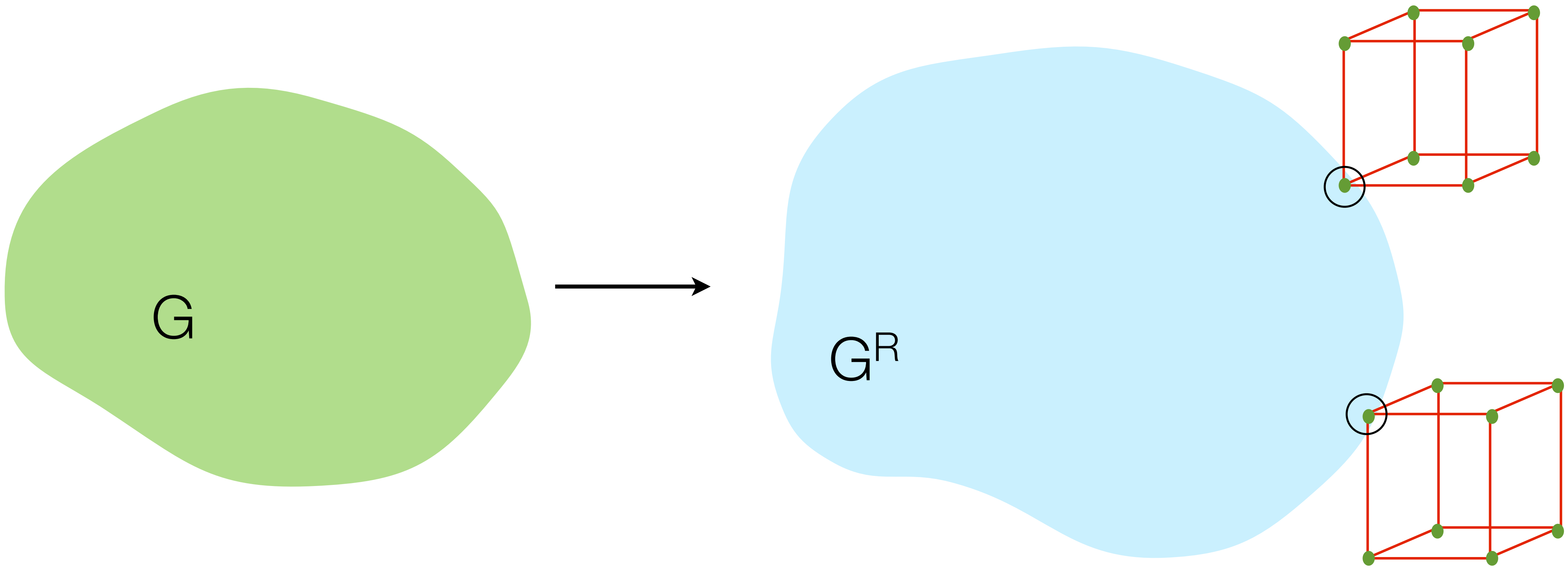
- Can still play the corresponding Unique Game. Also, balanced cut with expansion O(ϵ + η) by choosing all tuples which have a unique S-vertex.
- Modification only uses the fact that the strategy is about membership in a set.

# The resulting graph



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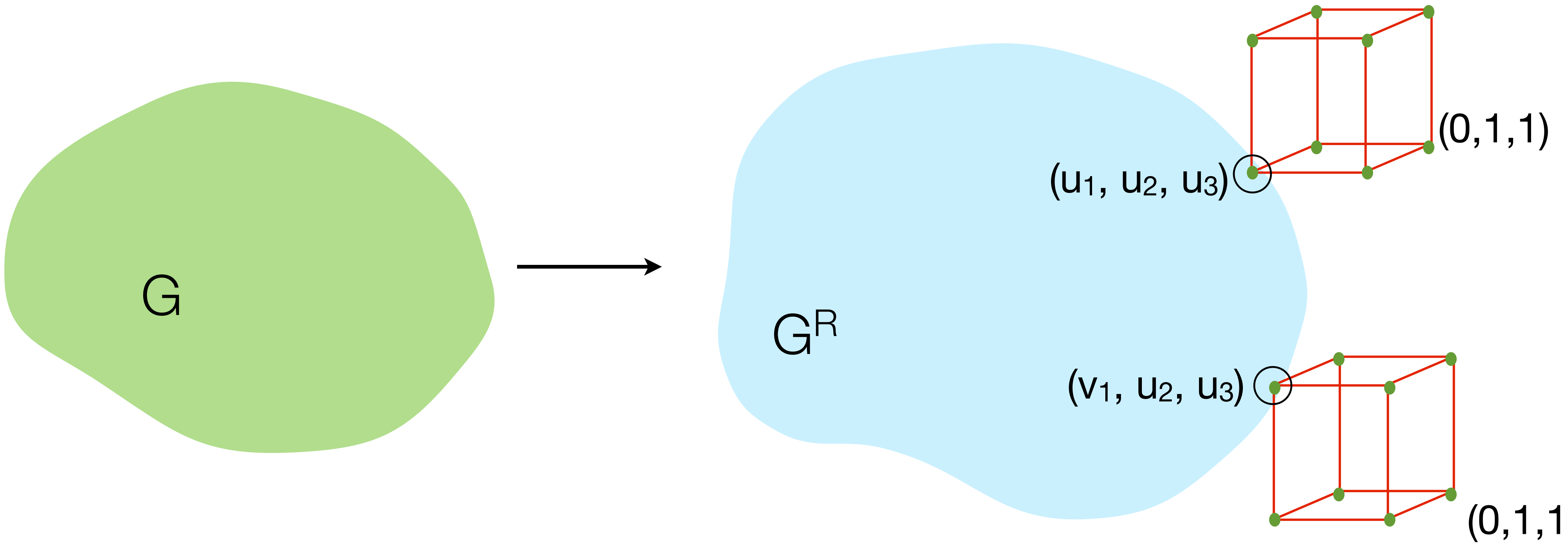

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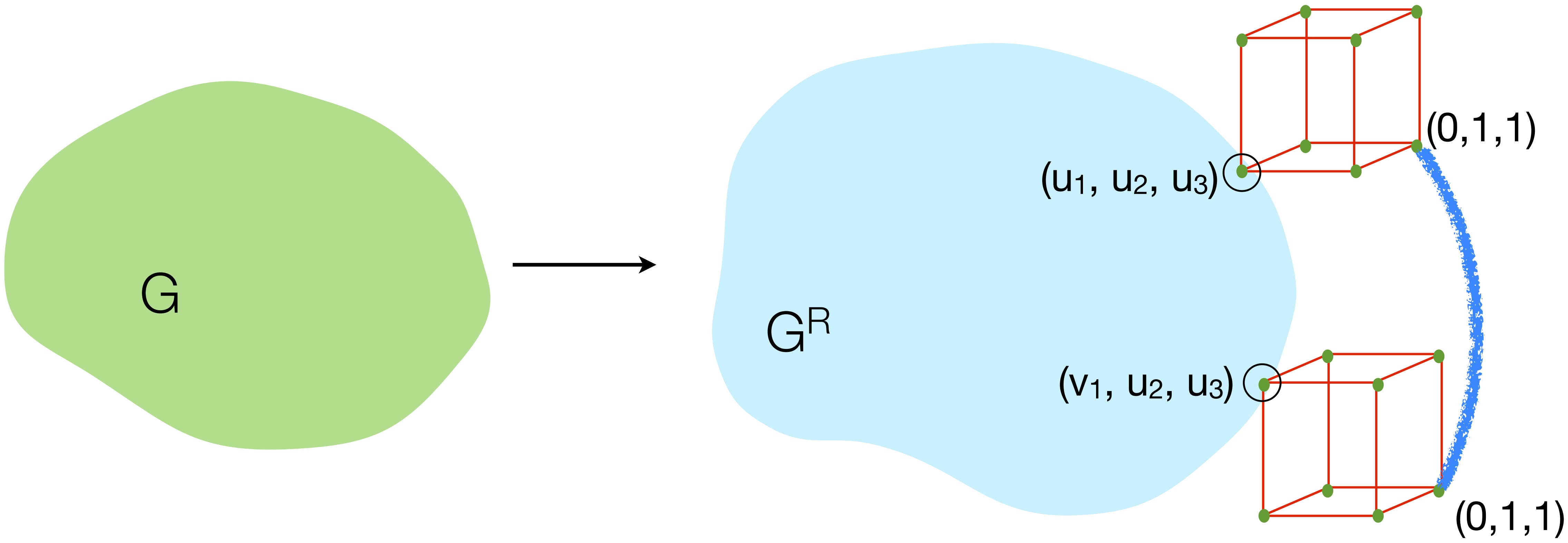

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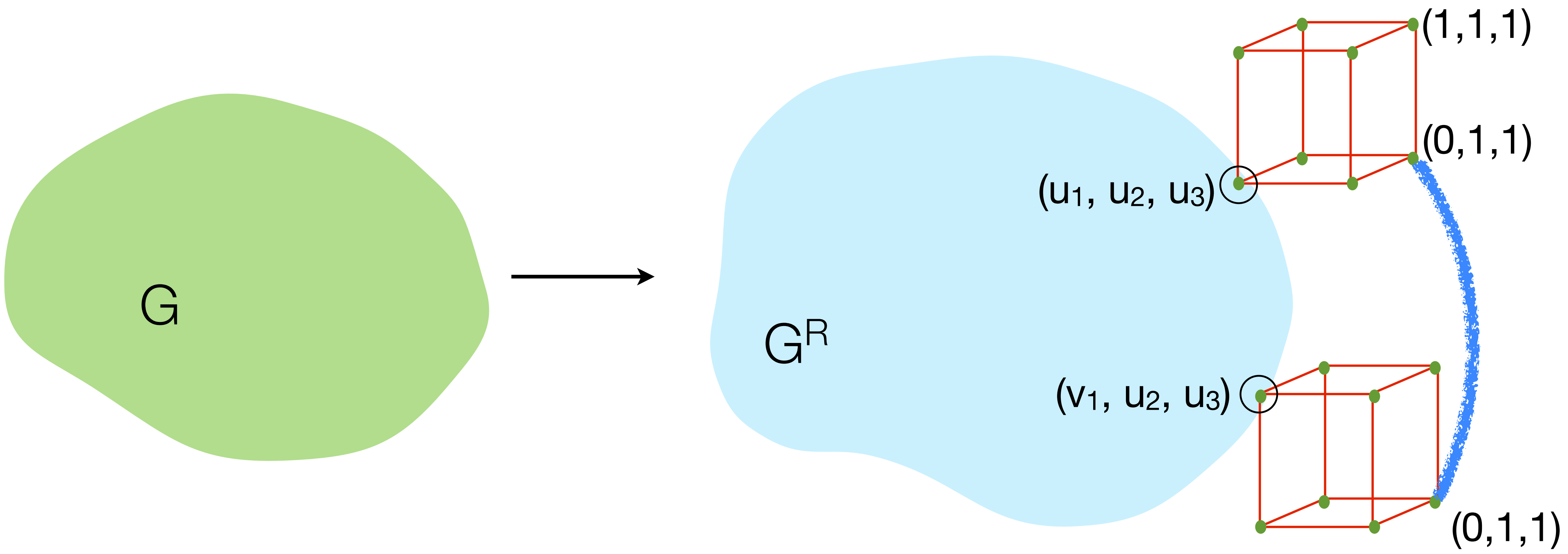


- $((u_1, u_2, u_3), (0,1,1))$ and $((v_1, u_2, u_3), (0,1,1))$ both become $((\perp, u_2, u_3), (0,1,1))$. Not so for (1,1,1).

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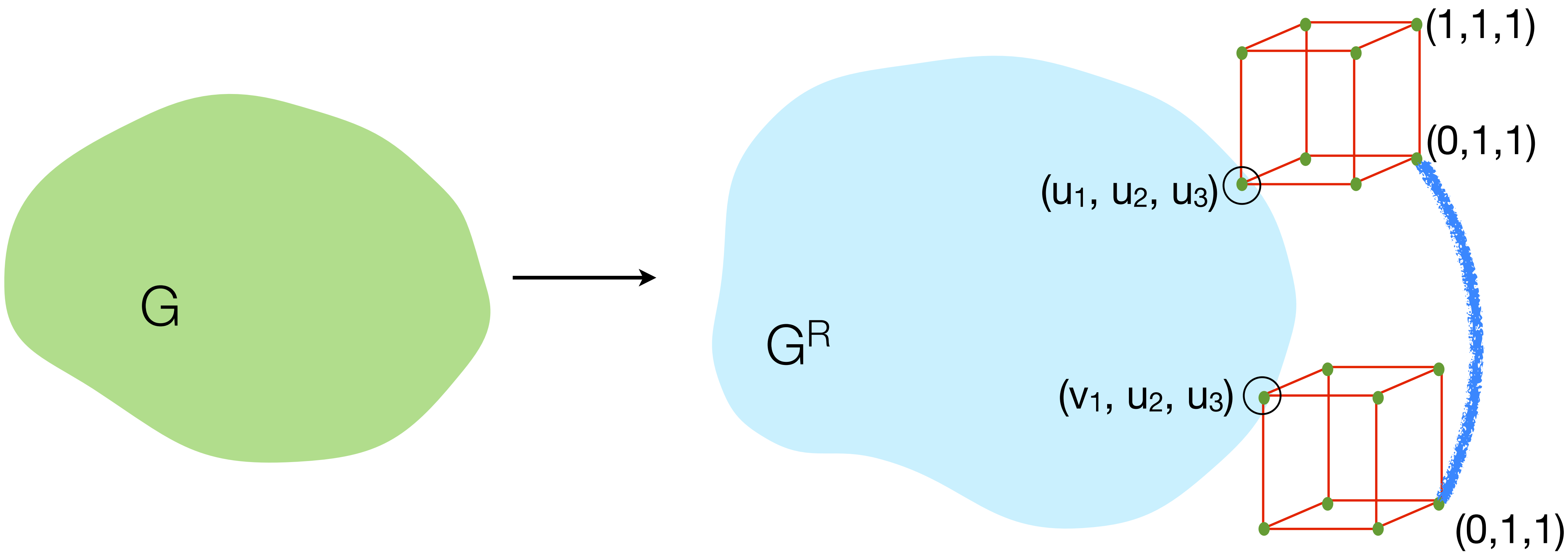


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- Completely destroys the gadget-based nature of the reduction.

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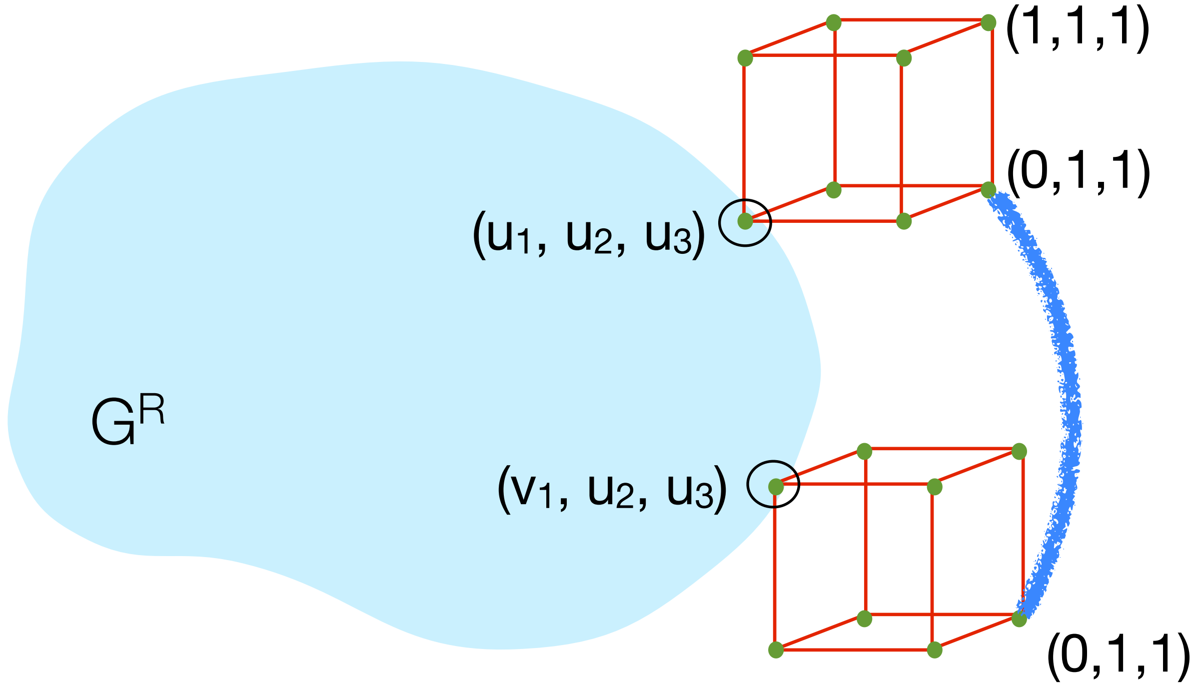

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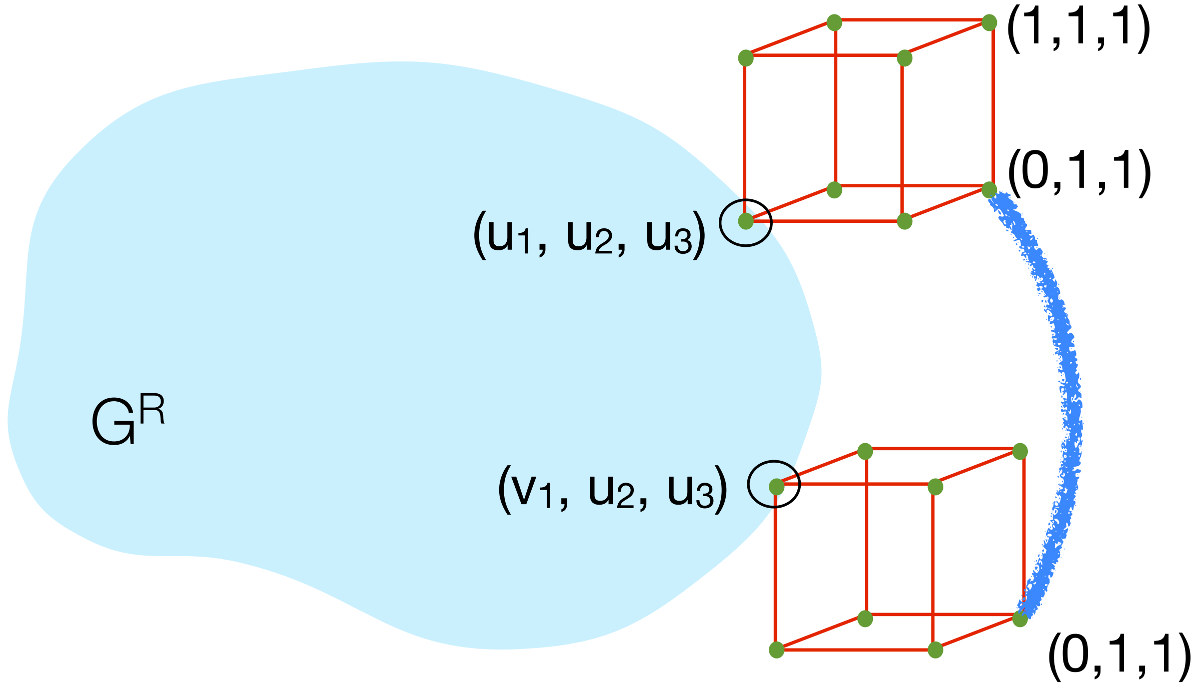


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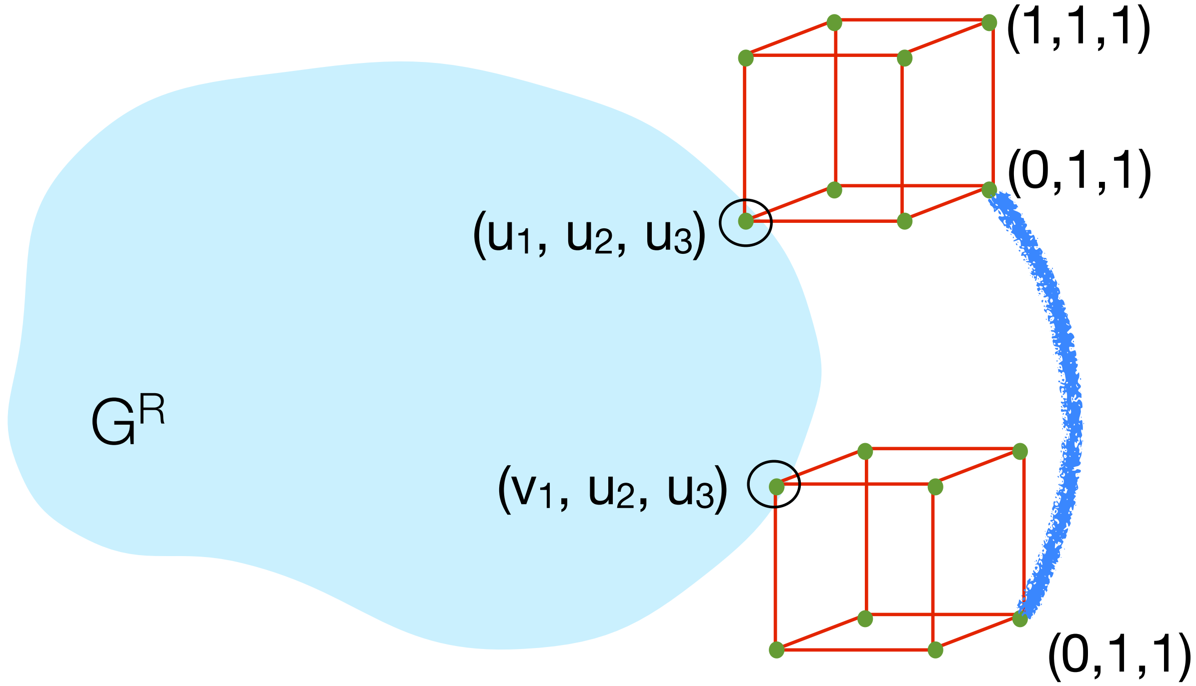


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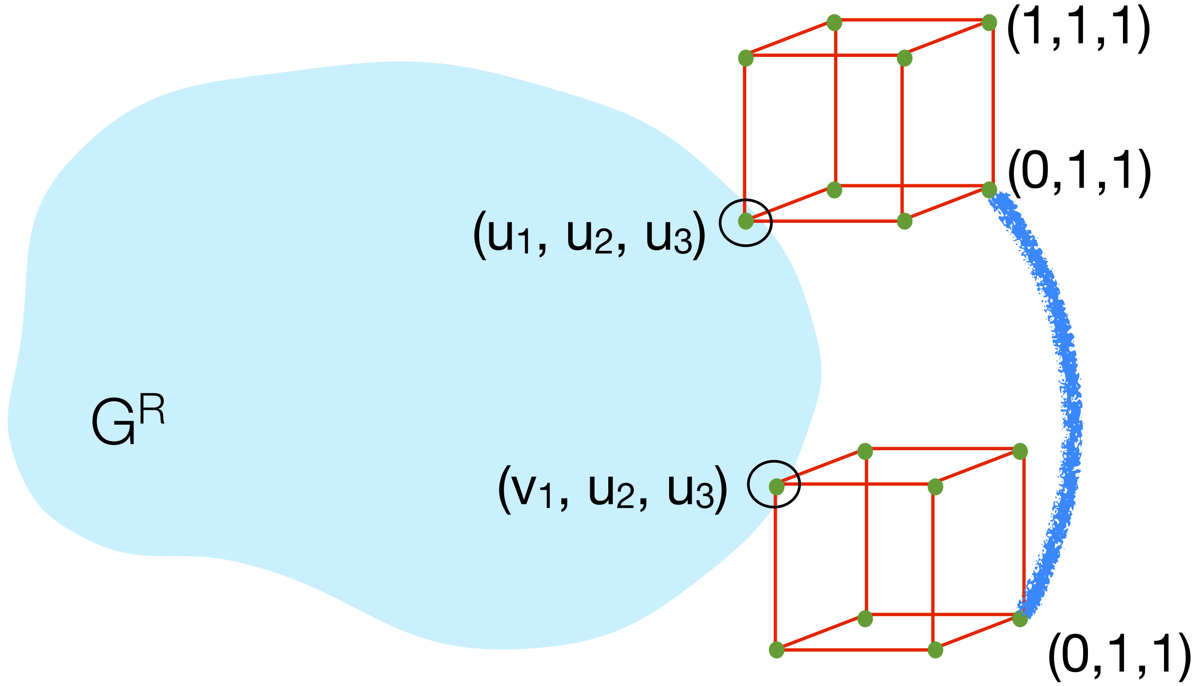


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- Obtain full result by using q-ary cubes and more general folding.

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- Can the folding operation be generalized to other classes of Unique Games where deleting part of the question still preserves a Unique Game.

- Reduction between scales in the other direction (Balanced Separator to SSE)?

Thank You

Questions?