TTIC/CMSC 31150 Mathematical Toolkit **Spring 2013**

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We know that given the adjacency matrix A of a d-regular graph, the eigenvalues of A have the relation

$$d = \mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_n = -d$$

and if $N = I - \frac{1}{d}$A is the normalized Laplacian then the eigenvalues of N are

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq 2$$

where $\lambda_i = 1 - \frac{1}{d}\mu_i$.

1 Quadratic form of a matrix

Let $L = dI - A$. For $x \in \mathbb{R}^n$,

$$x^T N x = \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2$$

$$\lambda_1 = \min_{x \in \mathbb{R}^n} \frac{x^T N x}{x^T x}$$

is the Rayleigh Quotient of x, denoted by R(x) and

$$\lambda_2 = \min_{x \in \mathbb{R}^n, x \perp \mathbf{1}} R(x)$$

If $\lambda_2 = 0$ then G has at least 2 connected components.

2 Cheeger's Inequality

Let $S \subseteq V$, $\varphi(S) = \frac{|E(S,\overline{S})|}{d|S|} = \mathbb{P}_{random(i,j) \in E}(j \in \overline{S}, i \in S)$
where $|E(S,\overline{S})|$ = number of edges going out of S
(for general graphs, $\varphi(S) = \frac{|E(S,\overline{S})|}{\sum deg(i)}$ where $\sum deg(i)$ is also known as the "volume of the graph").
and $\Phi_G = \min_{|S| \leq \frac{n}{2}} \varphi(S)$ be the expansion of the graph.

Then Cheeger's inequality says: $\frac{\lambda_2}{2} \leq \Phi_G \leq \sqrt{2\lambda_2}$

Proof:

Proof 1: To proove: $\frac{\lambda_2}{2} \leq \Phi_G$

Let $(S, \overline{S})$ be a cut in the graph s.t. $|S| \leq \frac{n}{2}$ and $\varphi(S) \leq \beta$. Then $\exists x \in \mathbb{R}^n, x \perp \mathbf{1}$ s.t. $R(x) \leq 2\beta$.

Consider the vector $\frac{1}{|S|}$ from S and the vector $\frac{-1}{|\overline{S}|}$ from $\overline{S}$. Then $\sum x_i = |S|.\frac{1}{|S|} + |\overline{S}|.(\frac{-1}{|\overline{S}|}) = 0$ i.e. orthogonal to $\mathbf{1}$.

Now $R(x) = \frac{1}{d}\frac{\sum\limits_{(i,j)\in E}(x_i - x_j)^2}{x^T x}$.

For any edge in $S, x_i = x_j$ so contribution in the sum $= 0$. Similarly for $|\overline{S}|$. Therefore the only contribution is that of edges going across which is $= \frac{1}{d}\frac{|E(S,\overline{S})|.(\frac{1}{|S|} + \frac{1}{|\overline{S}|})^2}{|S|.\frac{1}{|S|^2} + |\overline{S}|.\frac{1}{|\overline{S}|^2}} = \frac{1}{d}\frac{|E(S,\overline{S})|}{|S|}.\frac{|S| + |\overline{S}|}{|\overline{S}|} \leq$ $\beta.2$ i.e. 2β. Hence proved.

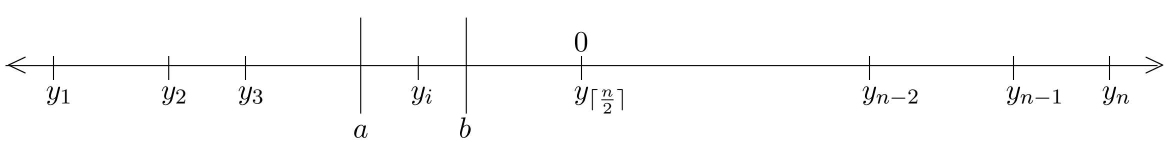
Proof 2: To proove: $\varphi_G \leq 2\sqrt{\lambda_2}$ (slightly worse bound)

Claim 2.1 *Given $x \perp \mathbf{1}$, we can find S s.t. $|S| \leq \frac{n}{2}$ and $\varphi_G \leq 2\sqrt{R(x)}$.*

Proof: Our proof is algorithmic (a linear time algorithm known as spectral partitioning)

Observation 1:
Multiplying x by a constant c does not change R(x).
Adding c to x:
Let y = x + c.$\mathbf{1}$. Then $R(y) \leq R(x)$ This is because: $y^T N y = x^T N x$ (numerators are same) and $\|y\|^2 = \|x + c\mathbf{1}\|^2 = \|x\|^2 + cn$ since x is orthogonal to $\mathbf{1}$ i.e. denominator of LHS is greater than that of RHS. Thus R(y) $\leq$ R(x).

Choose a c such that $y_{\lceil \frac{n}{2} \rceil} = 0$ i.e. let the median be at 0. We know that x gives a real number for each vertex on the number line. We cut this line.



Here a and b are cuts and there are n such cuts possible.

Claim 2.2 *One of these cuts is good enough for us.*

Proof: We notice that every edge will be looked at only twice. If we know the value for cut a, then to find the value for the next cut b, we only need to look at what happens to edges of y_i i.e. the vertex between a and b.

To avoid having to remember which side of the cut is $\leq \frac{n}{2}$, let
$\forall i, z_i^{(1)} = \max\{y_i, 0\}$ i.e. the positive part of y and
$z_i^{(2)} = \min\{y_i, 0\}$ i.e. the negative part of y. At least one of $z^{(1)}$ and $z^{(2)}$ must be non-zero.

Fact 2.3 $(z^{(1)})^T(z^{(1)}) + (z^{(2)})^T(z^{(2)}) = y^T y$

Fact 2.4 $(z^{(1)})^T N(z^{(1)}) \leq y^T N y$

This is because LHS $= \frac{1}{d}\sum_{(i,j)\in E}(z_i^{(1)} - z_j^{(1)})^2$ whereas the RHS is just the sum of absolute values. Same argument applies for $(z^{(2)})^T N(z^{(2)})$. Combined with the earlier fact we get that at least one of $(z^{(1)})^T(z^{(1)})$ and $(z^{(2)})^T(z^{(2)})$ must be $\leq \frac{y^T y}{2}$. Therefore $R(z^{(1)})$ or $R(z^{(2)})$ must be $\leq 2R(y)$ ■

Given $z \in [-1,1]^n, \exists S \subseteq \{i : z_i \neq 0\}$ s.t. $\varphi(S) \leq \sqrt{2R(z)}$ We will apply this to $z^{(1)}$ and $z^{(2)}$.

- choose random number $t \in [0,1]$ with uniform distribution
- let $S = \{i : z_i^2 \geq t\}$, then $\mathbb{P}$(we choose the i^{th} vertex)$= z_i^2$

Claim 2.5 $\frac{\mathbb{E}\left[|E(S,\overline{S})|\right]}{d\,\mathbb{E}\left[|S|\right]} \leq \sqrt{2R(z)}$

if this is true then since LHS is a positive quantity,
$\mathbb{E}_t\left[|E(S,\overline{S})| - d\sqrt{2R(z)}|S|\right] \leq 0 \Rightarrow \exists$ t s.t. $|E(S,\overline{S})| - d\sqrt{2R(z)}|S| \leq 0$.

Proof: Proof of claim:

$$Let\ Y_i = \begin{cases} 1 & \text{if i} \in \text{S} \\ 0 & \text{otherwise.} \end{cases}$$

then $\mathbb{E}\left[|S|\right] = \sum_i \mathbb{E}\left[Y_i\right] = \sum_i z_i^2$

$$\mathbb{E}\left[|E(S,\overline{S})|\right] = \sum_{(i,j)\in E} |z_i^2 - z_j^2|$$

$$= \sum_{(i,j)\in E} |z_i - z_j||z_i + z_j|$$

$$\leq \sqrt{\sum_{(i,j)\in E} |z_i - z_j|^2}\sqrt{\sum_{(i,j)\in E} (z_i + z_j)^2} \qquad \text{(by Cauchy Schwartz)}$$

$$\leq \sqrt{\sum_{(i,j)\in E} |z_i - z_j|^2}\sqrt{\sum_{(i,j)\in E} (2z_i^2 + 2z_j^2)}$$

$$= \sqrt{\sum_{(i,j)\in E} |z_i - z_j|^2}\sqrt{2d \sum_{(i,j)\in E} z_i^2}$$

Therefore $\dfrac{\mathbb{E}\left[|E(S,\overline{S})|\right]}{d\,\mathbb{E}\left[|S|\right]} \leq \dfrac{1}{d}\dfrac{\sqrt{\sum\limits_{(i,j)\in E}|z_i - z_j|^2}\sqrt{2d\sum\limits_{(i,j)\in E} z_i^2}}{\sum\limits_{(i,j)\in E} z_i^2}$

$= \sqrt{R(z)}$ ■

We are getting 2 outside root, we will try to get it inside now.

To proove: $\exists S \subseteq V$ s.t. $|S| \leq \frac{n}{2}$ and $\varphi(S) \leq \sqrt{2R(y)}$.

Let $y_{\frac{n}{2}} = 0$ and $y_1^2 + y_n^2 = 1$. Choose $t \in [y_1, y_n]$ where t is a random variable with density $2|t|$ defined only between y_1 and y_n.

Let $S = \{i : y_i \leq t\}$. Then for $a \geq y_1$ and $b \leq y_n$,

$P(t \in [a,b]) = \int_a^b 2|r|\,dr$

$= sgn(b)\cdot b^2 - sgn(a)\cdot a^2$

Claim 2.6 $\dfrac{\mathbb{E}\left[|E(S,\overline{S})|\right]}{d\,\mathbb{E}\left[min(|S|,\overline{|S|})\right]} \leq \sqrt{2R(y)}$

After this proof same as above.

Proof of claim:

$\mathbb{E}\left[|E(S,\overline{S})|\right] = \sum\limits_{(i,j)\in E} P\left(t \in (y_i, y_j)\right)$

$= \sum\limits_{(i,j)\in E} |sgn(y_i)\,y_i^2 - sgn(y_j)\,y_j^2|$

$= \sum\limits_{(i,j)\in E} |sgn(y_i)\,y_i^2 - sgn(y_j)\,y_j^2|$

Claim 2.7 $|sgn(y_i)\,y_i^2 - sgn(y_j)\,y_j^2| \leq |y_i - y_j|(|y_i| + |y_j|)$

If this is true then we continue the same process as above.

Proof of claim:

If same signs: obvious

If different signs then it is $= |y_i - y_j|\,(|y_i| + |y_j|) - (|y_i| + |y_j|)^2$

Hence proved.

Let

$$Y_i = \begin{cases} 1 & \text{if } (i \in S \text{ and } |S| \leq \frac{n}{2}) \text{ OR } (\text{i} \in \overline{S} \text{ and } |\overline{S}| \leq \frac{n}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

i.e.

$$Y_i = \begin{cases} 1 & \text{if } y_i \leq t \leq 0 \text{ OR } y_i \geq t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}\left[min(|S|,|\overline{S}|)\right] = \sum\limits_i \mathbb{E}\left[Y_i\right]$

$= \sum_i y_i^2$

So $\frac{1}{d}\frac{\mathbb{E}\left[|E(S,\overline{S})|\right]}{\mathbb{E}\left[\underset{i}{min}(|S|,\overline{|S|})\right]}$

$\leq \frac{\sqrt{2\sum(y_i - y_j)^2}\sqrt{2d\sum_i {y_i}^2}}{d\sum_i {y_i}^2}$

$= \sqrt{2R(y)}$

Thus $\varphi_G \leq \sqrt{2\lambda_2}$

Hence Proved.

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3 Some generalizations of Cheeger's inequality

Here we found sets S_1, S_2 s.t. $S_2 = \overline{S_1}$, $\varphi(S_1) \leq \sqrt{2\lambda_2}$ and $\varphi(S_2) \leq \sqrt{2\lambda_2}$

We can also find sets $S_{1....}S_k$ that are disjoint s.t. $\forall i\ \varphi(S_i) \leq O(k^2)\sqrt{\lambda_k}$

Later results also showed that we can find $S_1, S_2....S_{\frac{k}{2}}$ s.t. $\varphi(S_i) \leq O(\sqrt{\lambda_k logk})$ (cannot do better than log k)

Dirichlet Boundary Conditions:

Find vectors $z^{(1)}, z^{(1)} \ldots z^{(k)}$ s.t. $\forall i, j \{i : z^{(i)} \neq 0\} \cap \{j : z^{(j)} \neq 0\} = \varphi$ and $\forall i, R(z^{(i)}) = O(k^4).\lambda_k$

How to find z_i?

Arrange the k eigenvectors in a matrix and read them row wise. We get vectors $\in \mathbb{R}^k$. Cluster them.