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1 Existence of eigenvalues

We shall complete the proof of the following proposition which we sketched in the last class:

Proposition 1.1 *Let V be a finite-dimensional inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then φ has at least one eigenvalue.*

Let us assume for now that V is an inner product space over $\mathbb{C}$. As was observed in class, in this case we don't need self-adjointness to guarantee an eigenvalue. We thus prove the following more general result

Proposition 1.2 *Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $\varphi : V \to V$ be a linear operator. Then φ has at least one eigenvalue.*

Proof: Let $\dim(V) = n$. Let $v \in V \setminus 0_V$ be any non-zero vector. Consider the set of $n+1$ vectors $\{v, \varphi(v), \ldots, \varphi^n(v)\}$. Since the dimension of V is n, there must exists $c_0, \ldots, c_n \in \mathbb{C}$ such that

$$c_0 \cdot v + c_1 \cdot \varphi(v) + \cdots + c_n \varphi^n(v) \; = \; 0_V \, .$$

We assume above that $c_n \neq 0$, otherwise we can only consider the sum to the largest i such that $c_i \neq 0$. Let $P(x)$ denote the polynomial $c_0 + c_1 x + \cdots + c_n x^n$. Then the above can be written as $(P(\varphi))(v) = 0$, where $P(\varphi) : V \to V$ is a linear operator defined as

$$P(\varphi) \; := \; c_0 \cdot \mathsf{id} + c_1 \cdot \varphi + \cdots + c_n \varphi^n \, ,$$

with id used to denote the identity operator. Since P is a degree-n polynomial over $\mathbb{C}$, it can be factored into n linear factors, and we can write $P(x) = c_n \prod_{i=1}^{n} (x - \lambda_i)$ for $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. This means that we can write

$$P(\varphi) \; = \; c_n (\varphi - \lambda_n \cdot \mathsf{id}) \cdots (\varphi - \lambda_1 \cdot \mathsf{id}) \, .$$

Let $w_0 = v$ and define $w_i = \varphi(w_{i-1}) - \lambda_i \cdot w_{i-1}$ for $i \in [n]$. Note that $w_0 = v \neq 0_V$ and $w_n = P(\varphi)(v) = 0_V$. Let i^* denote the largest index i such that $w_i \neq 0_V$. Then, we have

$$0_V \; = \; w_{i^*+1} \; = \; \varphi(w_{i^*}) - \lambda_{i^*+1} \cdot w_{i^*} \, .$$

This implies that w_{i^*} is an eigenvector with eigenvalue λ_{i^*+1}. ■

To prove Proposition 1.1 using this, we note that $\varphi = \varphi^*$ implies the eigenvalue found by the above proposition must we real.

Exercise 1.3 *Use the fact that the eigenvalues of a self-adjoint operattor are real to prove Proposition 1.1 even when V is an inner product space over $\mathbb{R}$.*

2 Rayleigh quotients: eigenvalues as optimization

Definition 2.1 *Let $\varphi : V \to V$ be a self-adjoint linear operator and $v \in V \setminus \{0_V\}$. The* Rayleigh quotient *of φ at v is defined as*

$$\mathcal{R}_\varphi(v) \quad := \quad \frac{\langle \varphi(v), v \rangle}{\|v\|^2}\,.$$

Proposition 2.2 *Let $dim(V) = n$ and let $\varphi : V \to V$ be a self-adjoint operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then,*

$$\lambda_1 \;=\; \min_{v \in V \setminus \{0_V\}} \mathcal{R}_\varphi(v) \quad \text{and} \quad \lambda_n \;=\; \max_{v \in V \setminus \{0_V\}} \mathcal{R}_\varphi(v)$$

Using the above, Rayleigh quotients can be used to prove the spectral theorem for Hilbert spaces, by showing that the above maximum is attained at a point in the space, and defines an eigenvalue if the operator φ is "compact". See linked notes on the course webpage for a proof.

Proposition 2.3 (Courant-Fischer theorem) *Let $dim(V) = n$ and let $\varphi : V \to V$ be a self-adjoint operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then,*

$$\begin{aligned}\lambda_k &= \max_{\substack{S \subseteq V \\ dim(S) = n-k+1}} \; \min_{v \in S \setminus \{0_V\}} \mathcal{R}_\varphi(v) \\ &= \min_{\substack{S \subseteq V \\ dim(S) = k}} \; \max_{v \in S \setminus \{0_V\}} \mathcal{R}_\varphi(v)\,.\end{aligned}$$

Definition 2.4 *Let $\varphi : V \to V$ be a self-adjoint operator. Φ is said to be* positive semidefinite *if $\mathcal{R}_\varphi(v) \geq 0$ for all $v \neq 0$. Φ is said to be* positive definite *if $\mathcal{R}_\varphi(v) > 0$ for all $v \neq 0$.*

Proposition 2.5 *Let $\varphi : V \to V$ be a self-adjoint linear operator. Then the following are equivalent:*

1. *$\mathcal{R}_\varphi(v) \geq 0$ for all $v \neq 0$.*
2. *All eigenvalues of φ are non-negative.*
3. *There exists $\alpha : V \to V$ such that $phi = \alpha^* \alpha$.*

The decomposition of a positive semidefinite operator in the form $\varphi = \alpha^* \alpha$ is known as the Cholesky decomposition of the operator.