Convex Optimization

Problem set 3

Due Wednesday, April 14th

1. Consider the log-barrier method for solving an optimization problem with a single semi-definite constraint:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_1(x) \preccurlyeq 0 \\ & Ax = b \end{aligned} \tag{1}$$

where $f_1 : \mathbb{R}^n \to S^k$, $A \in \mathbb{R}^{n\times p}$ and $b \in \mathbb{R}^p$. That is, the semi-definite constraint is on a $k \times k$ matrix. We assume here the problem is feasible.

In order to solve the above problem, we will use the barrier problems:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^n} \quad & f_0(x) - \frac{1}{t}\log\det(-f_1(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned} \tag{2}$$

Let $x^*(t)$ and $\nu^*(t)$ be primal and dual optimal solutions of (2) and set $\lambda^*(t) = -\frac{1}{t}x^*(t)^{-1}$.

(a) Prove that $x^*(t)$ is feasible for (1) and $\lambda^*(t), \nu^*(t)$ is feasible for its dual, attaining the dual objective:

$$g(\lambda^*(t), \nu^*(t)) = f_0(x^*(t)) - \frac{k}{t}$$

(b) Conclude that $x^*(t)$ is $\frac{k}{t}$-suboptimal for (1).

(c) Consider a problem with multiple semi-definite constraints:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \preccurlyeq 0, \quad i = 1, \dots, m \\ & Ax = b \end{aligned} \tag{3}$$

where $f_i : \mathbb{R}^n \to S^{k_i}$, $A \in \mathbb{R}^{n\times p}$ and $b \in \mathbb{R}^p$. The associated barrier problems are:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^n} \quad & f_0(x) - \frac{1}{t}\sum_{i=1}^{m}\log\det(-f_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned} \tag{4}$$

Let $x^*(t)$ be an optimum of (4) and prove that it is $\frac{1}{t}\sum_{i=1}^{m} k_i$-suboptimal for (3) (Hint: reduce the problem to a problem with a single semi-definite constraint, and be sure to show that the barrier problems remain unchanged).

2. Boyd and Vandenberghe Problem 10.2

3. Boyd and Vandenberghe Problem 11.6