Homework Assignment 3

TTIC 31010 and CMSC 37000-1

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Problem 1. (Problem 3 from HW 2)
Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and non-negative edge capacities $\{c(e)\}$. Let $f : E \to \mathbb{R}_{\geq 0}$ be a maximum flow in G. Let G_f be the residual graph. Denote by S the set of nodes reachable from s in G_f and by T the set of nodes from which t is reachable in G_f. That is,

$$S = \{u : \text{ there is a directed path from } s \text{ to } u \text{ in } G_f\},$$
$$T = \{v : \text{ there is a directed path from } v \text{ to } t \text{ in } G_f\}.$$

Prove that $V = S \cup T$ if an only if G has a *unique* s-t minimum cut (i.e. an s-t cut whose capacity is strictly less than the capacity on any other s-t cut).

Problem 2. Let G be a flow network, $\{c(e)\}$ be a set of edge capacities, and $f : E \to \mathbb{R}_{\geq 0}$ be a maximum flow in the network G with capacities $c(e)$. Consider another set of capacities $\{c'(e)\}$ s.t.

- $c'(e) = c(e)$ for every edge e saturated by f (i.e. for every $e \in E$ s.t. $f(e) = c(e)$),
- $c'(e) \geq c(e)$ for every edge $e \in E$.

Prove that f is a maximum flow in G with capacities $\{c'(e)\}$.

Problem 3. Consider a bipartite graph $G = (X \cup Y, E)$. Suppose that G is a k regular graph; that is, the degree of every vertex equals k.

- Prove that G has a perfect matching.
- Prove that moreover there are k disjoint perfect matchings in G. That is, there are perfect matchings $M_1, \ldots, M_k$ s.t. $M_i \cap M_j = \varnothing$ for every $i \neq j$.

Problem 4. In this question we will help a hospital figure out whether it has enough supplies for blood transfusions for its patients. There are x_A patients with blood type A, x_B patients with blood type B, x_{AB} patients with blood type AB, and x_O patients with blood type O currently at the hospital, and each patient needs a transfusion of one unit of blood. The hospital has at its disposal s_A units of blood of type A, s_B of type B, s_{AB} of type AB and s_O of type O. The rules of blood transfusion are as follows:

- Patients with blood type A can receive only blood of types A or O.
- Patients with blood type B can receive only blood types B or O.
- Patients with blood type O can receive only blood of type O.
- Patients with blood type AB can receive any of the four types.

Design an efficient algorithm that determines whether the hospital's blood supply is sufficient for treating the patients, and if so, computes a way to distribute the hospital supplies among the patients, so that each of them receives blood of an appropriate type.