Homework Assignment 4

TTIC 31010 and CMSC 37000-1

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Problem 1. We will design an algorithm that finds a minimum cut in a directed graph using linear programming (LP). Suppose that we are given a directed graph $G = (V, E)$, source s, sink t and a set of edge capacities $\{c(e)\}$. In class, we showed that the dual to the Maximum Flow linear program is a linear program for the Minimum Cut Problem. Specifically, we derived the following linear program for the Minimum Cut Problem:

$$\text{minimize:} \quad \sum_{e \in E} c(e) y_e$$

subject to

$$d_u - d_v + y_{(u,v)} \geq 0 \quad \text{for every edge } (u, v) \in E$$

$$d_s = 0$$

$$d_t = 1$$

$$y_e \geq 0 \quad \text{for every edge } e \in E$$

Then we showed that for every s-t cut (A, B) there is a corresponding feasible solution (d, y) whose value equals the capacity of the cut (A, B). We concluded, using LP duality, that the solution that corresponds to a minimum cut is an optimal solution for this linear program. In this problem, we will develop an algorithm that given any feasible solution (d, y) finds a cut whose capacity is less than or equal to the LP value $\sum_{e \in E} c(e) y_e$.

Consider an arbitrary feasible solution (d, y). Let $p \in [0, 1)$. Define $A_p = \{u : d_u \leq p\}$ and $B_p = \{u : d_u > p\}$.

1. Prove that (A_p, B_p) is an s-t cut.

2. Consider the following function (where $(u, v) \in E$ is an edge)

$$I_{(u,v)}(x) = \begin{cases} 1, & \text{if } d_u \leq x < d_v \\ 0, & \text{otherwise} \end{cases}$$

   Prove that the capacity of the cut (A_p, B_p) equals $\sum_{e \in E} c(e) I_e(p)$.

3. Prove that for every edge $e \in E$, we have $\int_0^1 I_e(x) dx \leq y_e$.

4. Let $g(x) = \text{cap}(A_x, B_x)$. Prove that $\int_0^1 g(x)dx \le \sum_{e\in E} c(e)y_e$.

5. Conclude that for some $x \in (0,1)$, $\text{cap}(A_x, B_x) = g(x) \le \sum_{e\in E} c(e)y_e$.

6. Prove that $g(x)$ is a piece-wise constant function. Find all discontinuities of $g(x)$. Design an efficient algorithm that finds x at which $g(x)$ attains its minimum in $[0,1)$.

7. Design an efficient algorithm that given an optimal solution to the linear program for the Minimum Cut Problem, returns a minimum cut.

Definition 1. Suppose that we are given an underected graph $G = (V, E)$. A *vertex cover* is a subset of vertices C such that every edge of G is incident to at least one vertex in C. A minimum vertex cover is a vertex cover of smallest possible size. The Minimum Vertex Cover Problem asks to find a minimum vertex cover.

Problem 2. In this problem, we will design an algorithm for solving the Minimum Vertex Cover Problem in bipartite graphs. Let $G = (X \cup Y, E)$ be a bipartite graph with parts X and Y. Consider the following linear programming formulation of the Minimum Vertex Cover Problem. There is an LP variable x_u for every vertex $u \in X \cup Y$.

$$
\begin{aligned}
\text{minimize:} \quad & \sum_{u\in X\cup Y} x_u \\
\text{subject to} \quad & \\
& x_u + x_v \ge 1 \quad \text{for every edge } (u,v) \in E \\
& x_u \ge 0
\end{aligned}
$$

1. Prove that the value of this linear program is at most the minimum vertex cover size.

2. Let $\hat{x}$ be an optimal solution. Prove that $\hat{x}_u \in [0,1]$ for every $u \in X \cup Y$.

3. Assume additionally that $\hat{x}$ is a vertex of the feasible polytope. Prove that then $\hat{x}_u \in \{0,1\}$. To this end, consider the set of vertices A:

$$A = \{u \in X \cup Y : x_u \ne 0 \text{ and } x_u \ne 1\}.$$

   We need to prove that A is empty. Assume to the contrary that A is not empty. Denote $\varepsilon = \min\{\hat{x}_u, 1 - \hat{x}_u : u \in A\}$. Consider solutions x' and x'' defined by

$$
x'_u = \begin{cases} \hat{x}_u, & \text{if } u \notin A \\ \hat{x}_u + \varepsilon, & \text{if } u \in A \cap X \\ \hat{x}_u - \varepsilon, & \text{if } u \in A \cap Y \end{cases}
\qquad
x''_u = \begin{cases} \hat{x}_u, & \text{if } u \notin A \\ \hat{x}_u - \varepsilon, & \text{if } u \in A \cap X \\ \hat{x}_u + \varepsilon, & \text{if } u \in A \cap Y \end{cases}
$$

   Prove that x' and x'' are feasible solutions. Conclude that x is not a vertex of the feasible polytope.

4. Design an algorithm that solves the Minimum Vertex Cover problem in bipartite graphs. The algorithm may use a subroutine that finds an optimal solution $\hat{x}$ that is a vertex of the feasible polytope.

Problem 3. Write the dual to the linear program for the Minimum Vertex Cover problem in bipartite graphs. What problem does this linear program represent?