Tutorial 8: Linear Programming

TTIC 31010 and CMSC 37000-1

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Problem 1. In this problem, we will compute the shortest path between two vertices using linear programming. Suppose that we are given a directed graph $G = (V, E)$, two vertices $s, t \in V$, and a length function $\ell : E \to \mathbb{R}_{>0}$. Let $d(u, v)$ be the length of the shortest path between u and v in G w.r.t. ℓ. We introduce an LP variable x_u for every vertex u other than s. Consider the following linear program:

$$\begin{array}{lll} \text{maximize:} & x_t & \\ \text{subject to} & x_v - x_u \leq \ell(u, v) & \text{for every edge } (u, v) \in E, \text{s.t. } u \neq s, v \neq s \\ & x_v \leq \ell(s, v) & \text{for every edge } (s, v) \in E \\ & x \geq 0 & \end{array}$$

Prove that the optimal LP value equals $d(s, t)$. Specifically, do the following.

- Prove that the optimal LP value is at least $d(s, t)$. To this end, construct a feasible solution whose value equals $d(s, t)$.
- Prove that the optimal LP value is at most $d(s, t)$.

Problem 2. Write the dual to the LP from Problem 1. Suppose that P is a shortest path between s and t. Construct a feasible solution to the dual whose value equals the length of P (i.e. that equal $d(s, t)$).