Outer Bi-Lipschitz Extensions & Dimension Reductions

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Overview

• Kirszbraun theorem & Lipschitz extensions
• Outer bi-Lipschitz extensions
  • definitions
  • results
• Applications to dimension reduction
  • terminal dimension reductions
  • prioritized dimension reductions
• Proofs
Kirszbraun Theorem

Let $A \subset \ell_2^m$ and $f : A \to \ell_2^n$ be Lipschitz. There is a Lipschitz extension $\tilde{f} : \ell_2^m \to \ell_2^n$ s.t.

$$\|\tilde{f}\|_{Lip} = \|f\|_{Lip}$$
Kirszbraun for bi-Lipschitz maps?

Now, let $f : A \to \ell^m_2$ be bi-Lipschitz. Is there a bi-Lipschitz extension $\tilde{f} : \ell^m_2 \to \ell^n_2$ s.t.

$$D(\tilde{f}) \approx D(f)$$
Kirszbraun for bi-lipschitz maps?

No!

• \( \mathbb{R}^2 \to \mathbb{R} \). Let \( A = \langle e_1 \rangle \) and \( f: xe_1 \mapsto x \).

There is no bi-Lipschitz embedding of \( \mathbb{R}^2 \) into \( \mathbb{R} \).
Kirszbraun for bi-lipschitz maps?

No!

\[ f \text{ maps } 0, 1, 2 \text{ to } 0, 2, 1, \text{ respectively. Is there } \tilde{f}: \mathbb{R} \rightarrow \mathbb{R}? \]

There is no \textit{continuous injective extension}, let alone a bi-Lipschitz extension.
Kirszbraun for bi-Lipschitz maps?

Let’s use extra dimensions!

\( \mathbb{R}^2 \to \mathbb{R} \). Let \( A = \langle e_1 \rangle \) and \( f : xe_1 \mapsto x \).

Assume that target \( \mathbb{R} \subset \mathbb{R}^2 \). Then \( \tilde{f} = id_{\mathbb{R}^2 \to \mathbb{R}^2} \).
Kirszbraun for bi-Lipschitz maps?

Let’s use extra dimensions!

\[ \mathbb{R} \to \mathbb{R}. \ f \text{ maps } 0, 1, 2 \text{ to } 0, 2, 1, \text{ respectively.} \]
Outer bi-lipschitz extension

Given $A \subset \ell_2^m$ and bi-Lipschitz $f : A \to \ell_2^n$

Identify $\ell_2^n$ with subspace $\langle e_1, ..., e_n \rangle \subset \ell_2^N$

\[ \tilde{f} : \ell_2^m \to \ell_2^N \] is an outer bi-Lipschitz extension of $f$ if

- $\tilde{f}(a) = f(a) \oplus 0 \equiv f(a)$ for every $a \in A$
- $\tilde{f}$ is bi-Lipschitz
Outer bi-lipschitz extension

\( \tilde{f} : \ell^m_2 \to \ell^N_2 \) is an outer bi-Lipschitz extension of \( f \) if
\[
\tilde{f}(a) = f(a) \oplus 0 \quad \text{for every} \quad a \in A
\]
and \( \tilde{f} \) is bi-Lipschitz.

**Theorem.** For every bi-Lipschitz \( f : A \to \ell^m_2 \), there exists an outer bi-Lipschitz extension with
\[
D(\tilde{f}) \leq 3D(f).
\]

[Alestalo, Väisälä ‘97] \( D(\tilde{f}) \leq c \, D(f)^2 \)
Near Isometric Case

Near isometric case: Assume that

\[ D(f) = 1 + \varepsilon \]

If \( \varepsilon = 0 \), i.e., \( f \) is an isometric embedding, there is an isometric extension \( \tilde{f} \)

Is there a bi-Lipschitz extension with

\[ D(\tilde{f}) = 1 + \delta_\varepsilon \]

where \( \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) ?

\[ \delta_\varepsilon \geq \frac{c}{\log^2 1/\varepsilon} \]
Highly nonlinear map \( f : \mathbb{R} \rightarrow \ell^2_2 \)

Let \( f(x) \) is a point with polar coordinates \((x, c \log x)\) for \( x > 0 \); \( f(-x) = -f(x) \).
Near Isometric Case: 1-point extension

Theorem.

Let $A \subset \ell_2^m$ and $f: A \to \ell_2^n$, and $v \in \ell_2^m$.
Suppose $D(f) = 1 + \varepsilon$.

There exists an outer extension $\tilde{f}: A \cup \{v\} \to \ell_2^{n+1}$

$$D(\tilde{f}) = 1 + c\sqrt{\varepsilon}$$

The bound is tight.
**Near Isometric Case: \( \mathbb{R} \rightarrow \mathbb{R} \)**

**Theorem.**

Let \( A \subset \mathbb{R} \) and \( f: A \rightarrow \mathbb{R} \). Suppose \( D(f) = 1 + \epsilon \).

Then there is an outer extension \( \tilde{f}: \mathbb{R} \rightarrow \mathbb{R} \)

\[
D(\tilde{f}) = 1 + \frac{c}{\log^2 1/\epsilon}
\]

The bound is tight.
Near Isometric Case

Assume that $D(f) = 1 + \varepsilon$. Is there a bi-Lipschitz extension with

$$D(\tilde{f}) = 1 + \delta_{\varepsilon}$$

where $\delta_{\varepsilon} \to 0$ as $\varepsilon \to 0$?

$$\delta_{\varepsilon} \geq \frac{c}{\log^2 1/\varepsilon}$$
Applications

Dimension Reductions
**Terminal Dimension Reduction**

[Elkin, Filtser, and Neiman ‘17]

Let $T \subset \ell_2^n$ be a set of points called **terminals**.

A terminal dimension reduction $f : \ell_2^n \rightarrow \ell_2^d$ is a map s.t.

$$\| f(p) - f(v) \| \approx \| x - v \|$$

to a factor of $D$ for every $p \in T$ and $v \in \ell_2^n$

$f$ approximately preserves distances between terminals and between terminals and non-terminals.
Terminal Dimension Reduction

[EFN '17] There exists a terminal dimension reduction with distortion $C$ and $d = \log |T|$.

[MMMR '18] For every $\varepsilon > 0$, there exists a terminal dimension reduction with distortion $1 + \varepsilon$ and

$$d = \frac{\log |T|}{\varepsilon^4}.$$

Upcoming talk [Narayanan, Nelson '19] gets the optimal dependence on $\varepsilon$. 
Terminal Dimension Reduction

Proof: Let \( f : T \to \ell^d_2 \) be a dimension reduction with distortion \( 1 + \varepsilon^2 \) [Johnson–Lindenstrauss ‘84].

Use the 1-point bi-Lipschitz extension theorem to extend it to every point \( v \in \ell^n_2 \setminus T \).

Obtain \( \tilde{f}(v) \equiv v' \in \ell_2^{d+1} \) s.t.

\[
\|f(p) - v'\| \approx \|p - v\|
\]

up to a factor of \( 1 + c\sqrt{\varepsilon^2} = 1 + c\varepsilon \) for every \( p \in T \).
Two Stage Dimension Reduction

\[ T \subset X \subset \ell_2^n \]

There exists dimension reduction \( f \) s.t.

- \( f(T) \subset \ell_2^{d_1} \) where \( d_1 = \frac{c \log |T|}{\varepsilon^2} \)
- \( f(X) \subset \ell_2^{d_2} \) where \( d_2 = \frac{c \log |X|}{\varepsilon^2} \)

\[ \|f(x) - f(y)\| \approx \|x - y\| \]

up to \( 1 + \varepsilon \) for \( x, y \in T \)

up to \( 3 + \varepsilon \) for \( x, y \in X \)
Two Stage Dimension Reduction

\[ T \subset X \subset \ell_2^n \]

- First, let \( f : T \rightarrow \ell_2^{d_1} \) be a dimension reduction
- Extend \( f \) to \( X \), obtain \( \tilde{f} : X \rightarrow \ell_2^{d_1+n} \)
- Apply dimension reduction to extra coordinates

\[
(a_1, \ldots, a_{d_1}) \oplus (a_{d_1+1}, \ldots, a_n)
\]

\[
(a_1, \ldots, a_{d_1}) \oplus \pi(a_{d_1+1}, \ldots, a_n)
\]
Prioritized Dimension Reduction

Let \( X = \{x_1, \ldots, x_n\} \subset \ell_2^N \).

There exists a dimension reduction \( f: X \to \ell_2^{c \log^2 n} \) s.t.

- \( f(\{x_1, \ldots, x_k\}) \subset \ell_2^{\log^3 + \varepsilon k} \)
- \( f \) has distortion \( \sim \log \log k \) on \( \{x_1, \ldots, x_k\} \)

Use \( \sim \log \log \log \log n \) stages.

The notion of a prioritized embedding was introduced by Elkin, Filtser, and Neiman (2017).
Proofs
Outer bi-lipschitz extension

Consider \( f : A \to \ell^n_2 \) with

\[
\| u - v \| \leq \| f(u) - f(v) \| \leq D \| u - v \|
\]

- Kirszbraun \( \Rightarrow \) a \( D \)-Lipschitz extension \( f' : \ell^m_2 \to \ell^n_2 \)
- Want: construct \( g : \ell^m_2 \to \ell^m_2 \) and let \( \tilde{f} = f' \oplus g \)
  - \( g(a) = 0 \) for \( a \in A \)
  - \( g \) is \( cD \)-Lipschitz
  - \( \| f'(u) - f'(v) \| + \| g(u) - g(v) \| \geq \| u - v \| \)
Maps $f$ and $f^{-1}$
lipschitz ext. of $f$ and $f^{-1}$
$s = h \circ f'$ is $D$-Lipschitz

$\|s(u) - s(v)\| \leq \|f'(u) - f'(v)\|$

$D$-Lip $f'$

$1$-Lip $h$

$s(a) = a$ for $a \in A$
Outer bi-Lipschitz extension

Let $g(u) = u - s(u)$

- $g(a) = 0$ for $a \in A$ since
  $$g(a) = a - s(a) = 0$$

- $g(a)$ is $cD$-Lipschitz since $s$ is $D$-Lipschitz

$$\|f'(u) - f'(v)\| + \|g(u) - g(v)\| \geq \|f'(u) - f'(v)\| + \|u - v\| - \|s(u) - s(v)\| \geq \|u - v\|$$
1-point extension: notation

Given
\[ A = \{u_0, \ldots, u_k\} \subset \ell_2 \]
\[ f \text{ has distortion } 1 + \varepsilon \]
\[ v \in \ell_2 \]

Goal
Extend \( f \) to \( v \)
\[ v' = \tilde{f}(v) \]

Write
\[ x = y \pm \varepsilon \text{ if } |x - y| \leq c\varepsilon \]

- \( u_0 \) is the closest point to \( v \) in \( A \)
- \( u_0 = 0 \)
- \( \|v - u_0\| = \|v\| = 1 \)

\( \{\} \quad \text{WLOG} \)
1-point extension: notation

Given
\[ A = \{u_0, \ldots, u_k\} \subset \ell_2 \]
\[ f \text{ has distortion } 1 + \varepsilon \]
\[ v \in \ell_2 \]

Goal
Extend \( f \) to \( v \)
\[ v' = \tilde{f}(v) \]

WLOG

- \( u_1, \ldots, u_k \) are linearly independent
- Extend \( f \) to \( \text{span}(A) \) by linearity
- Get \( \varphi: \langle u_1, \ldots, u_k \rangle \to \ell_2^n \)
- \( \|\varphi\| \) may be arbitrarily large
From distances to inner products

\[ 2\langle u_i, u_j \rangle = \|u_i\|^2 + \|u_j\|^2 - \|u_i - u_j\|^2 \]

\[ 2\langle \varphi u_i, \varphi u_j \rangle = \|\varphi u_i\|^2 + \|\varphi u_j\|^2 - \|\varphi u_i - \varphi u_j\|^2 \]

Thus,

\[ \langle \varphi u_i, \varphi u_j \rangle = \langle u_i, u_j \rangle \pm \varepsilon \max\left(\|u_i\|^2, \|u_j\|^2\right) \]
From distances to inner products

- Let
  \[ w_i = \frac{u_i}{1 + \|u_i\|^2} \in B \]

- Then,
  \[ \langle \varphi w_i, \varphi w_j \rangle = \langle w_i, w_j \rangle \pm \varepsilon \]

\( \varphi \) approximately preserves inner products

- Goal #1: find \( v' \in B \) so that
  \[ \langle v', \varphi w_i \rangle = \langle v, w_i \rangle \pm \sqrt{\varepsilon} \]
From distances to inner products

Let $K = \text{conv}(\pm w_1, \ldots, \pm w_k)$

• Since $\langle \varphi w_i, \varphi w_j \rangle = \langle w_i, w_j \rangle \pm \varepsilon$,
  $\langle \varphi a, \varphi b \rangle = \langle a, b \rangle \pm \varepsilon$

for $a, b \in K$

• In particular,
  $\|\varphi a\| = \|a\| \pm \sqrt{\varepsilon}$

• Consider $a \in K$. Let $z = \frac{\varphi a}{\|\varphi a\|}$.

  $\langle \varphi a, z \rangle = \|\varphi a\| = \|a\| \pm \sqrt{\varepsilon} \geq \langle a, v \rangle - c\sqrt{\varepsilon}$
From distances to inner products

• \( \forall a \in K \exists z \in B \ \text{s.t.} \)
  \[ \langle \varphi a, z \rangle \geq \langle a, v \rangle - c\sqrt{\varepsilon} \]

• Minimax: \( \exists v' \in B \ \forall a \in K \)
  \[ \langle \varphi a, v' \rangle - \langle a, v \rangle \geq -c\sqrt{\varepsilon} \]

• Applies both to \( a \) and \(-a\)
  \[ |\langle \varphi a, v' \rangle - \langle a, v \rangle| \leq c\sqrt{\varepsilon} \]

• In particular,
  \[ |\langle \varphi w_i, v' \rangle - \langle w_i, v \rangle| \leq c\sqrt{\varepsilon} \]
From distances to inner products

$$|\langle \varphi w_i, v' \rangle - \langle w_i, v \rangle| \leq c \sqrt{\varepsilon}$$

• Make $v'$ unit vector

$$v'' = v' \oplus \sqrt{1 - \|v'\|^2}$$

extra coordinate

• We have

$$\|u_i - v\|^2 = \|u_i\|^2 + \|v\|^2 - 2\langle u_i, v \rangle$$

$$\|\varphi u_i - v''\|^2 = \|\varphi u_i\|^2 + \|v''\|^2 - 2\langle \varphi u_i, v'' \rangle$$
Summary

• Introduces the notion of outer bi-Lipschitz extension
• Proved an analogue of the Kirszbraun Theorem.
• Nearly isometric case $D = 1 + \varepsilon$
  • One-point extension with $D' = 1 + O(\sqrt{\varepsilon})$
  • Tight result in the one-dimensional case: $D' \sim \frac{1}{\log^2 1/\varepsilon}$
• Open problem: settle the general case

• Applications
  • Terminal dimension reductions with $D = 1 + \varepsilon$
  • Prioritized dimension reductions