Lipschitz and outer bi-Lipschitz extendability

Yury Makarychev, TTIC

Sepideh Mahabadi, Columbia ⇒ TTIC
Konstantin Makarychev, Northwestern
Ilya Razenshteyn, Microsoft Research

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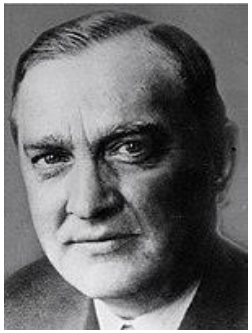
Plan

- The Lipschitz extendability problem
- Known results and open problems
- Vertex sparsifiers
- Connection between vertex sparsifiers and the Lipschitz extendability problem
- Outer bi-Lipschitz extendability: definition, results, and open problems

Hahn–Banach Theorem

H. Hahn



S. Banah

Let V be a normed space and $L \subset V$ be its linear subspace. Every bounded linear map

$$f : L \to \mathbb{R}$$

can be extended to $\tilde{f} : V \to \mathbb{R}$ so that

$$\|\tilde{f}\| = \|f\|$$

Is there an analogue for

- Lipschitz maps?
- maps into $\mathbb{R}^d$ or other normed spaces?

Preliminaries

$f: X \to Y$ is Lipschitz if for every $u, v \in X$

$$d_Y\big(f(u), f(v)\big) \le C d_X(u, v)$$

The Lipschitz constant $\|f\|_{Lip}$ of f is the minimum C s.t. that the inequality holds.

f is bi-Lipschitz if for some $C_1, C_2 > 0$, every $u, v \in X$

$$C_1 d_X(u, v) \le d_Y\big(f(u), f(v)\big) \le C_2 d_X(u, v)$$

The bi-Lipschitz constant or distortion $D(f)$ of f is the minimum of C_2/C_1 s.t. that the inequality holds.

Preliminaries

ℓ_p^d is $\mathbb{R}^d$ equipped with the $\|\cdot\|_p$ norm:

$$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$$

$$\|x\|_\infty = \max |x_i|$$

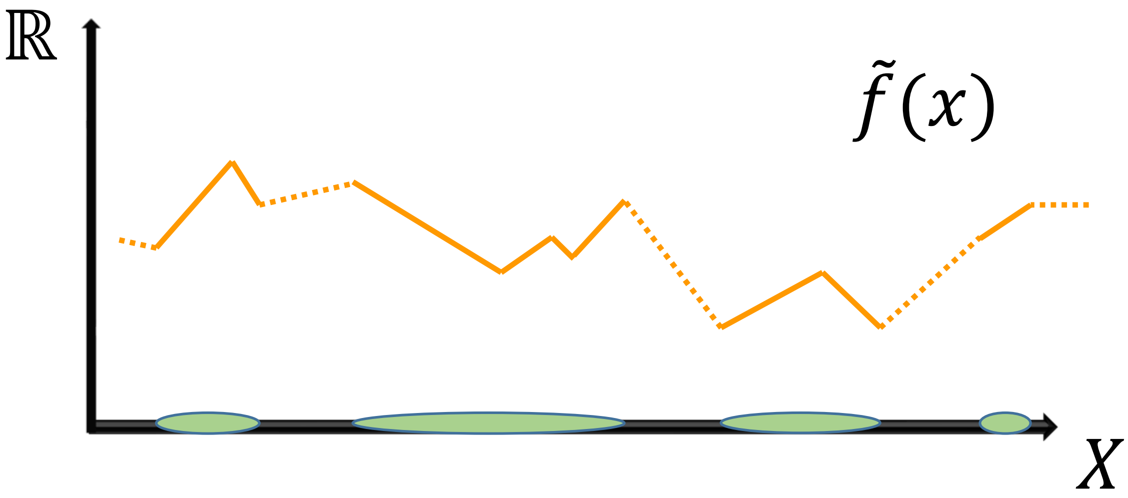
McShane–Whitney Theorem
"non-linear Hahn–Banach"

E. McShane

H. Whitney

Let (X, d) be a metric space and $A \subset X$. Every Lipschitz map $f: A \to \mathbb{R}$ can be extended to $\tilde{f}: X \to \mathbb{R}$ so that

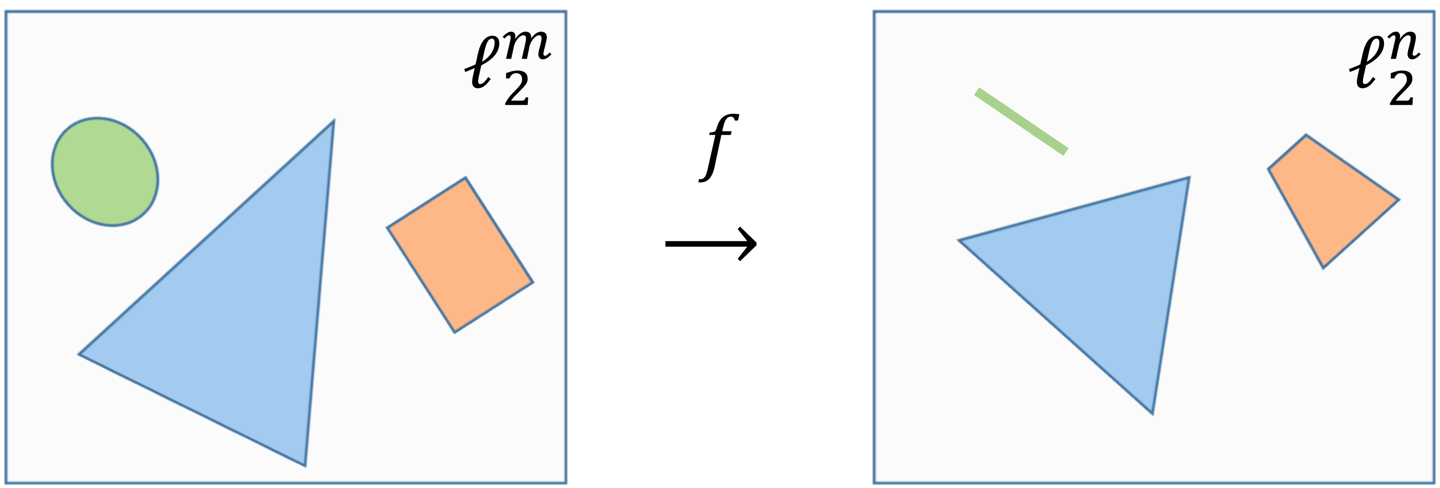
$$\|\tilde{f}\|_{Lip} = \|f\|_{Lip}$$



Kirszbraun Theorem

Let $A \subset \ell_2^m$. Every Lipschitz map $f: A \to \ell_2^n$ can be extended to $\tilde{f} : \ell_2^m \to \ell_2^n$ so that

$$\|\tilde{f}\|_{Lip} = \|f\|_{Lip}$$



Lipschitz Extension Constant

Let X be a metric space and Y be a normed space.

$e_k(X, Y)$ is the min C s.t. for every $A \subset X$ of size $\leq k$ and

$$f: A \to Y$$

there exists an extension $\tilde{f}: X \to Y$ with

$$\|\tilde{f}\|_{Lip} \leq C \|f\|_{Lip}$$

McShane–Whitney: $e_k(X, \mathbb{R}) = 1$

Kirszbraun: $e_k(\ell_2, \ell_2) = 1$

Lipschitz Extension Constant

In general, $e_k(X, Y) > 1$

E.g., $e_3\left(\ell_1^3, \ell_2^2\right) \geq \sqrt{3}$

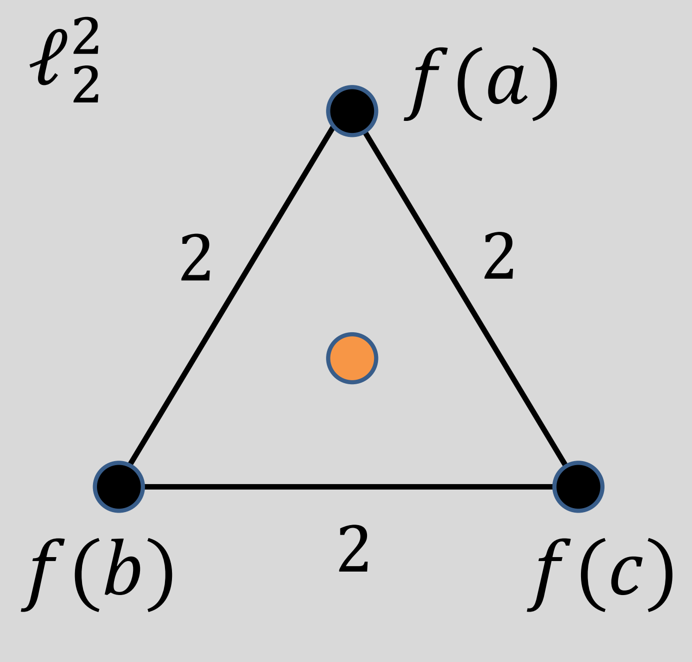
$A = \{a, b, c\}$

$a = (1,0,0)$

$b = (0,1,0)$

$c = (0,0,1)$

$d = (0,0,0)$



$X \to Y$	$e_k(X,Y)$	
any $\to \mathbb{R}$ or ℓ_∞	1	McShane, Whitney '34
$\ell_2 \to \ell_2$	1	Kirszbraun '34
$\ell_p \to \ell_2$ $p \leq 2$	$\leq C_p (\log k)^{\frac{1}{p}-\frac{1}{2}}$	Marcus, Pisier '84
$1 < p < 2$	$\geq c_p \left(\frac{\log k}{\log\log k}\right)^{\frac{1}{p}-\frac{1}{2}}$	Johnson, Lindenstrauss '84
any $\to \ell_2$	$\leq C\sqrt{\log k}$	JL '84
$\ell_p \to \ell_q$ $1 < q \leq 2 \leq p$	$\leq 24\sqrt{\frac{p-1}{q-1}}$	Naor, Peres, Schramm, Sheffield '04
other $p, q \in (1, \infty)$	$e_k \to \infty$ as $k \to \infty$	

Extension Results

Johnson–Lindenstruass–Schechtman '86

$$e_k(X,Y) \le C \log k$$

Lee–Naor '03

$$e_k(X,Y) \le \frac{C \log k}{\log \log k}$$

Best lower bounds are:

$$e_k \gtrsim c\sqrt{\log k}$$

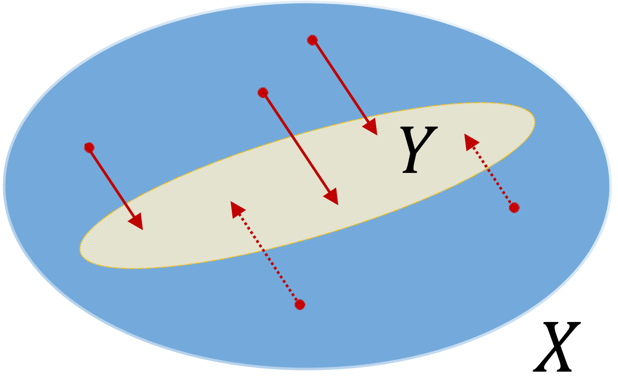
Open Problem: what is the dependence of e_k on k ?

[JL '84] Technique for proving lower bounds on $e_k(X,Y)$

Prove a lower bound for linear extensions

Reinterpret it as a lower bound for Lipschitz extensions

- Linear extension ("projection") constant is up to $\sqrt{\dim Y}$



- [JL '84] $e_k(X,Y) \geq c\sqrt{\frac{\log k}{\log\log k}}$.

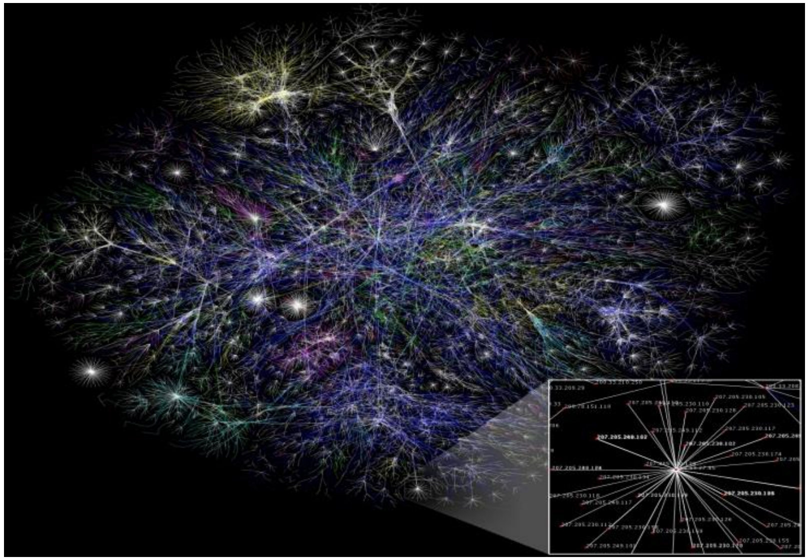
Open Problems

- Can the upper bound of $\sim \log k / \log\log k$ be improved?

- Are there any X and Y with $e_k(X, Y) \gg \sqrt{\log k}$?

- Ball '92: Is it true that $e_k(\ell_2, \ell_1) \leq C < \infty$?

Graph Sparsification

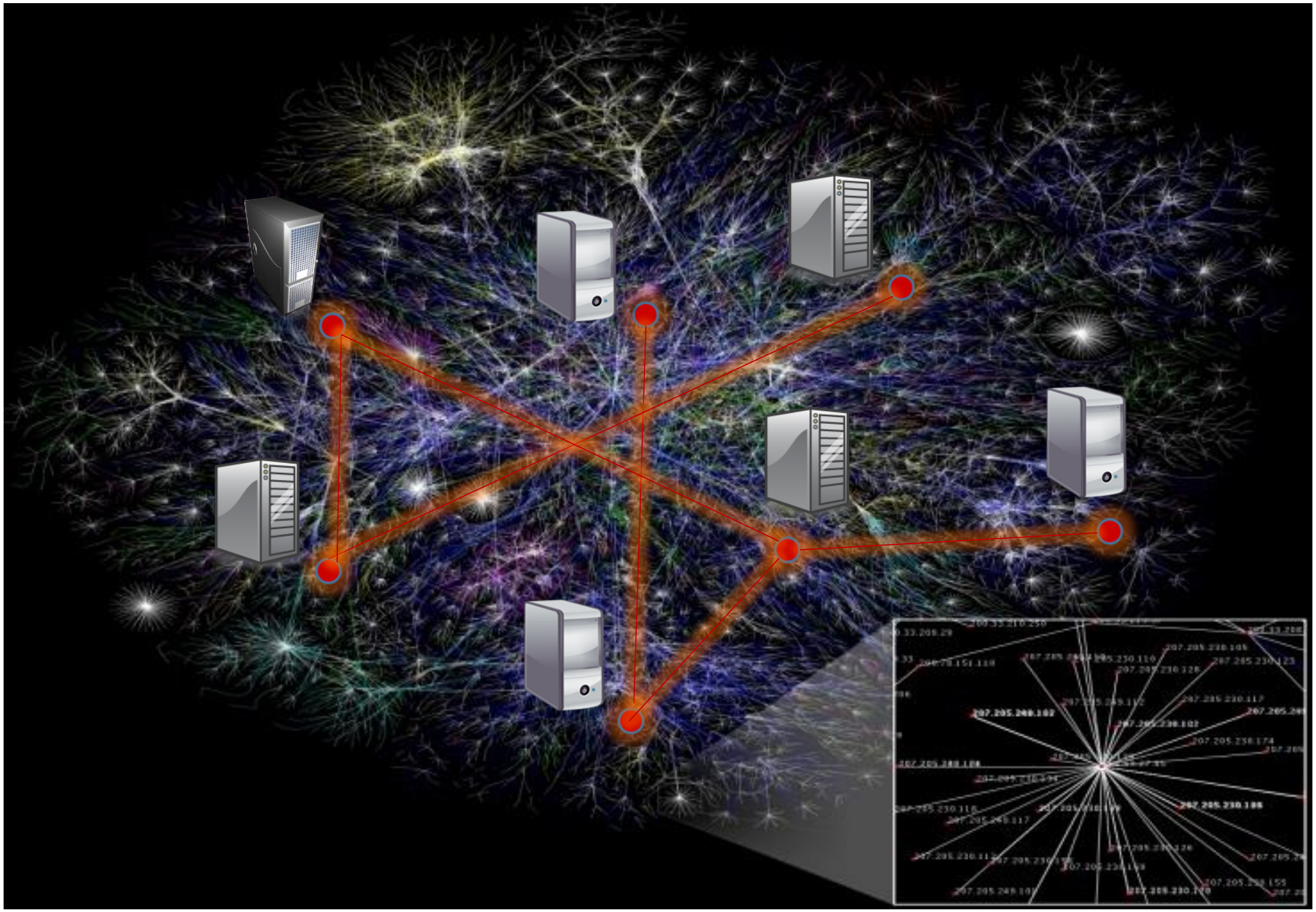


Given: a huge graph G

Goal: find a "simpler" graph H "similar" to G

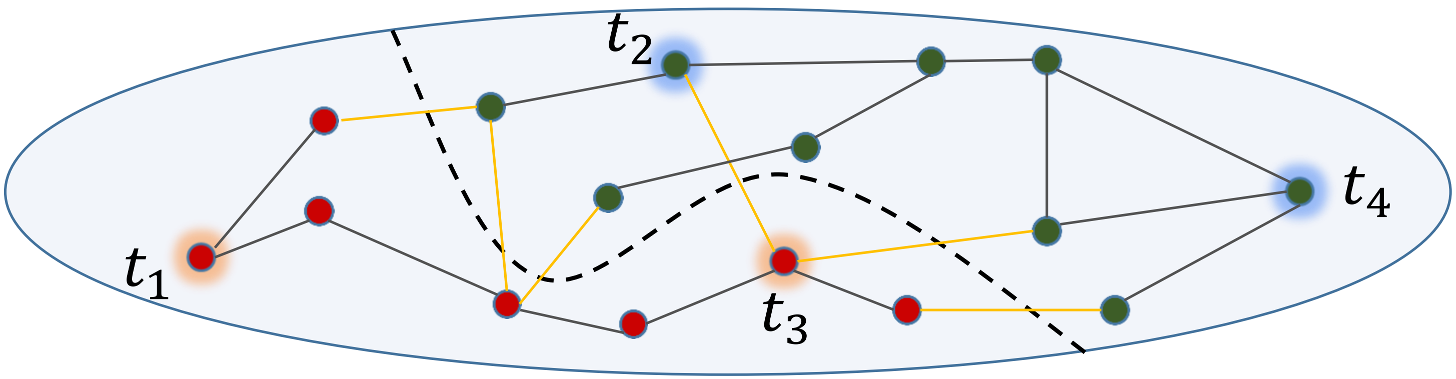
- compact representation
- algorithms work faster on the new graph
- can obtain better approximation results

Bottleneck & Routing Problems



Bottleneck Problem

- graph $G = (V, E)$ with edge capacities c_e
- set of terminals $T \subset V$

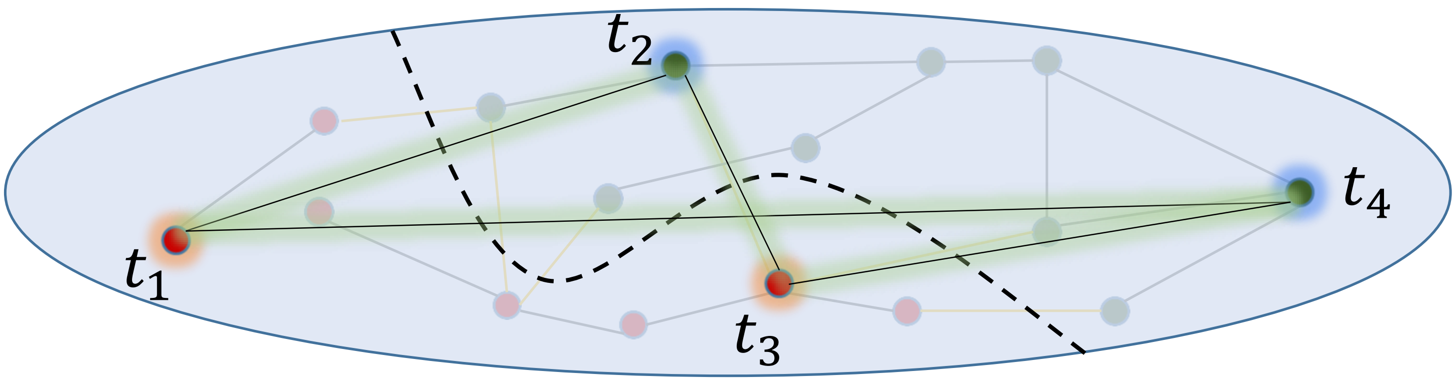


For $S \subset T, \mathrm{bk}_G(S)$ is the capacity of the minimum capacity cut in G that separates S and $T \setminus S$ in G.

Bottleneck Problem

[Moitra '09] Graph $H = (T, E')$ with capacities c'_e is a **vertex cut sparsifier** for G with distortion $D \geq 1$ if

$$\mathrm{bk}_G(S) \leq \mathrm{bk}_H(S) \leq D \cdot \mathrm{bk}_G(S) \qquad \forall S \subset T$$



Given H, can easily compute bottlenecks between terminals in the network!

Network Routing Problem

Routing problem: send a certain amount of data d_{ij} from each terminal t_i to t_j so that the total amount sent over each edge e is at most its capacity c_e.

[LM '10] A **vertex flow sparsifier** is an analogue of a vertex cut sparsifier for the network routing problem.

Known Results

Moitra '09 and Leigthon and Moitra '10

$$k = |T|$$

- $C \log k / \log\log k$ existential upper bound
- $C \log^2 k / \log\log k$ algorithmic upper bound
- $C > 1$ lower bound for cut sparsifiers
- $\Omega(\log\log k)$ lower bound for flow sparsifiers

Open Questions:

- $C \log k / \log\log k$ algorithmic upper bound?
- Better lower bounds?
- Better upper bounds?

Papers on Vertex Sparsification

Charikar, Leighton, Li and Moitra '10

Englert, Gupta, Krauthgamer, Räcke, Talgam and Talwar '10

Makarychev and Makarychev '10

Main Results

- Define "Metric Sparsifiers"
- Give $C \log k / \log \log k$ algorithmic upper bound [independently, CLLM '10, EGKRTT '10]
- Establish a direct connection between Vertex Sparsifiers and Lipschitz Extendability

$$Q_k^{cut} = e_k(\ell_1, \ell_1)$$

$$Q_k^{flow} = e_k(\ell_\infty, \ell_\infty \oplus_1 \ldots \oplus_1 \ell_\infty)$$

Lower bounds via Lipschitz Extendability

Using lower bounds for "projection constants" [Grünbaum '60], we get

$$Q_k^{flow} \geq e_k(\ell_\infty, \ell_1) \geq C\sqrt{\log k / \log \log k}$$

Figiel, Johnson, and Schechtman '88 implies

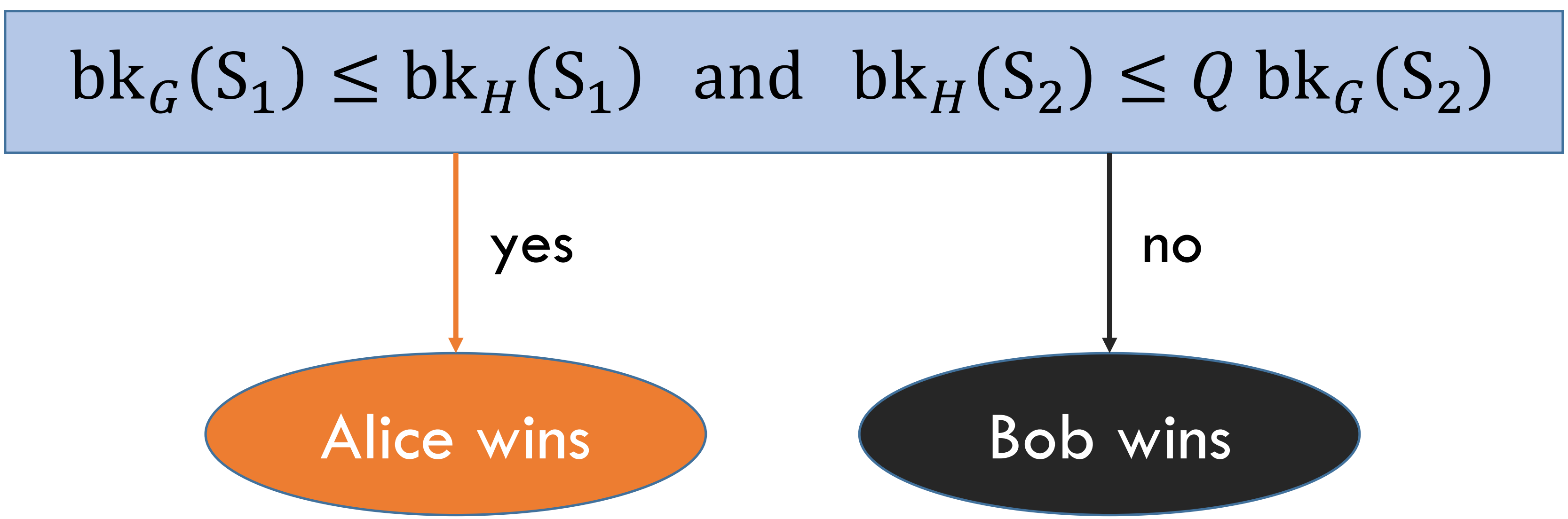
$$Q_k^{cut} = e_k(\ell_1, \ell_1) \geq \frac{C\sqrt{\log k}}{\log \log k}$$

Proof Idea: $Q_k^{cut} \le Q \equiv e_k(\ell_1, \ell_1)$

Consider a game: G and $\{c_e\}$ are fixed

Alice: defines H by providing c'_e

Bob: presents $(S_1, T \setminus S_1)$ and $(S_2, T \setminus S_2)$



Proof Idea: $Q_k^{cut} \leq Q \equiv e_k(\ell_1, \ell_1)$

Consider a game: G and $\{c_e\}$ are fixed

Bob: distribution of $(S_1, T \setminus S_1)$ and $(S_2, T \setminus S_2)$

Alice: defines H by providing c'_e

$\mathbb{E}\mathrm{bk}_G(\mathrm{S}_1) \leq \mathbb{E}\mathrm{bk}_H(\mathrm{S}_1) \quad \mathbb{E}\mathrm{bk}_H(\mathrm{S}_2) \leq Q\ \mathbb{E}\mathrm{bk}_G(\mathrm{S}_2)$

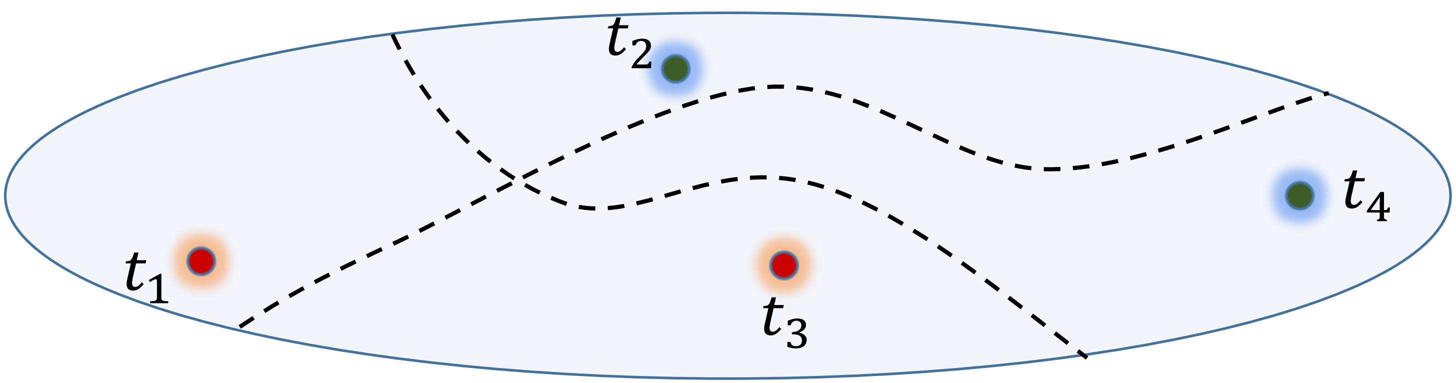
yes → Alice wins

no → Bob wins

Distribution of cuts

Distribution $\mathcal{D}$ of cuts $(S, T \setminus S)$ on T defines a map $f: T \to L_1(\Omega, \mu)$:

$$f(u) = \begin{cases} 0, & \text{if } x \in S \\ 1, & \text{if } x \notin S \end{cases}$$



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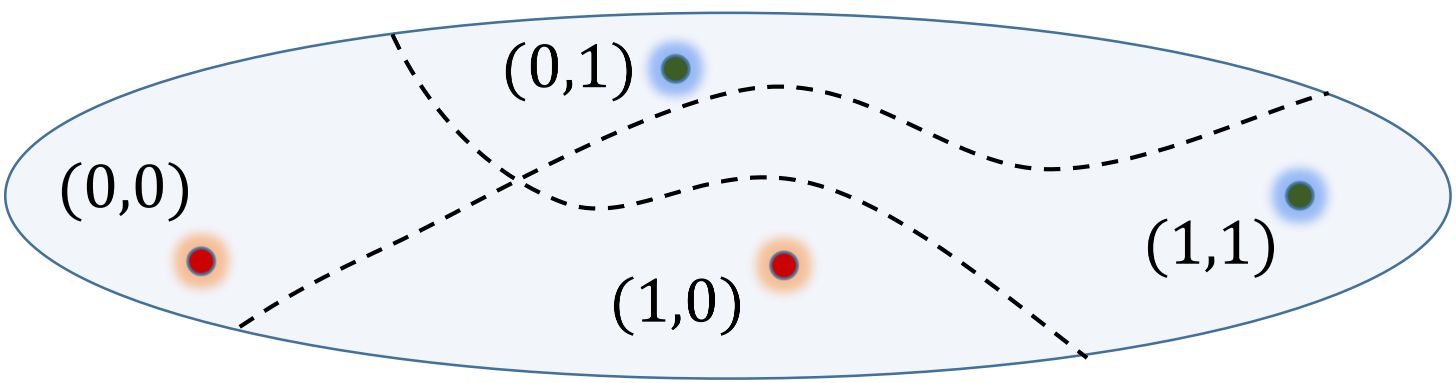


Distribution of cuts

$$f(u) = \begin{cases} 0, & \text{if } x \in S \\ 1, & \text{if } x \notin S \end{cases}$$

$$\Pr[u, v \text{ are separated by } S] = \|f(u) - f(v)\|_1$$

$$\mathbb{E}[\mathrm{bk}_H(S)] = \sum_{u,v \in T} c'(u,v) \cdot \|f(u) - f(v)\|_1$$

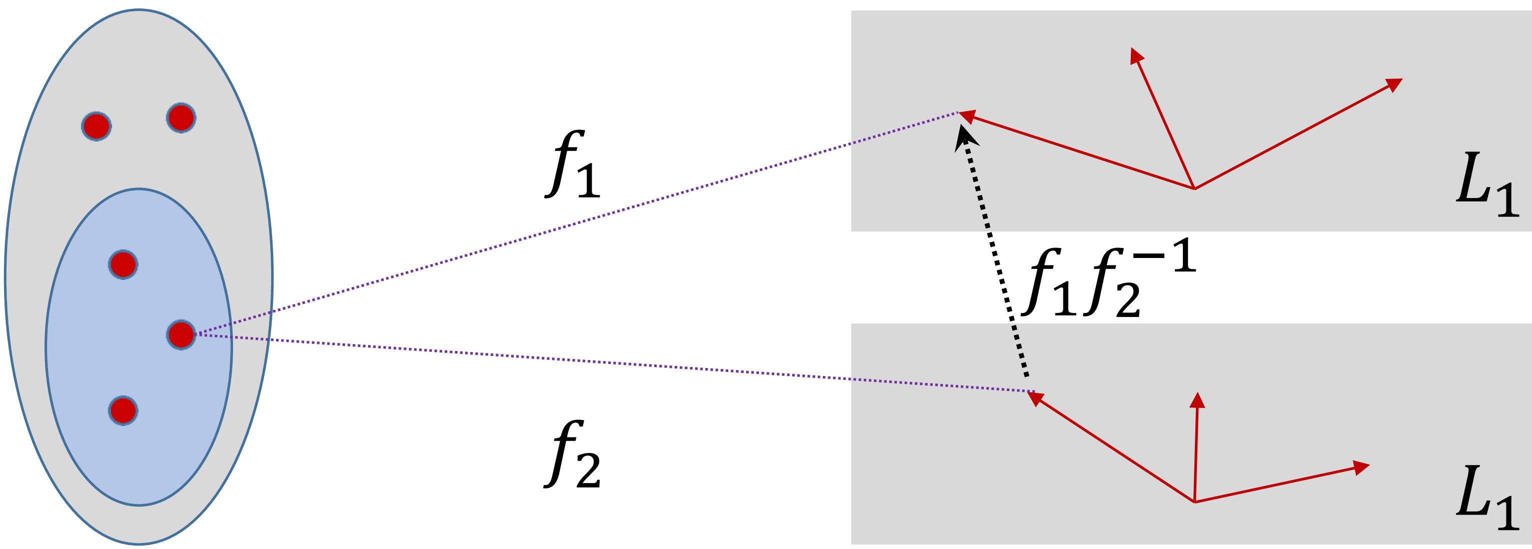
$$\mathbb{E}[\mathrm{bk}_G(S)] = \min_{\tilde{f}} \sum_{u,v \in V} c'(u,v) \cdot \left\|\tilde{f}(u) - \tilde{f}(v)\right\|_1$$

Bob: gives maps $f_1\colon T \to L_1$ and $f_2\colon T \to L_1$

Need:

$$\mathrm{bk}_G(\widetilde{f_1}) \le \mathrm{bk}_H(f_1)$$
$$\mathrm{bk}_H(f_2) \le Q\,\mathrm{bk}_G(\widetilde{f_2})$$

Relate $f_1 f_2^{-1}$ and $\widetilde{f_1}\,\widetilde{f_2}^{-1}$

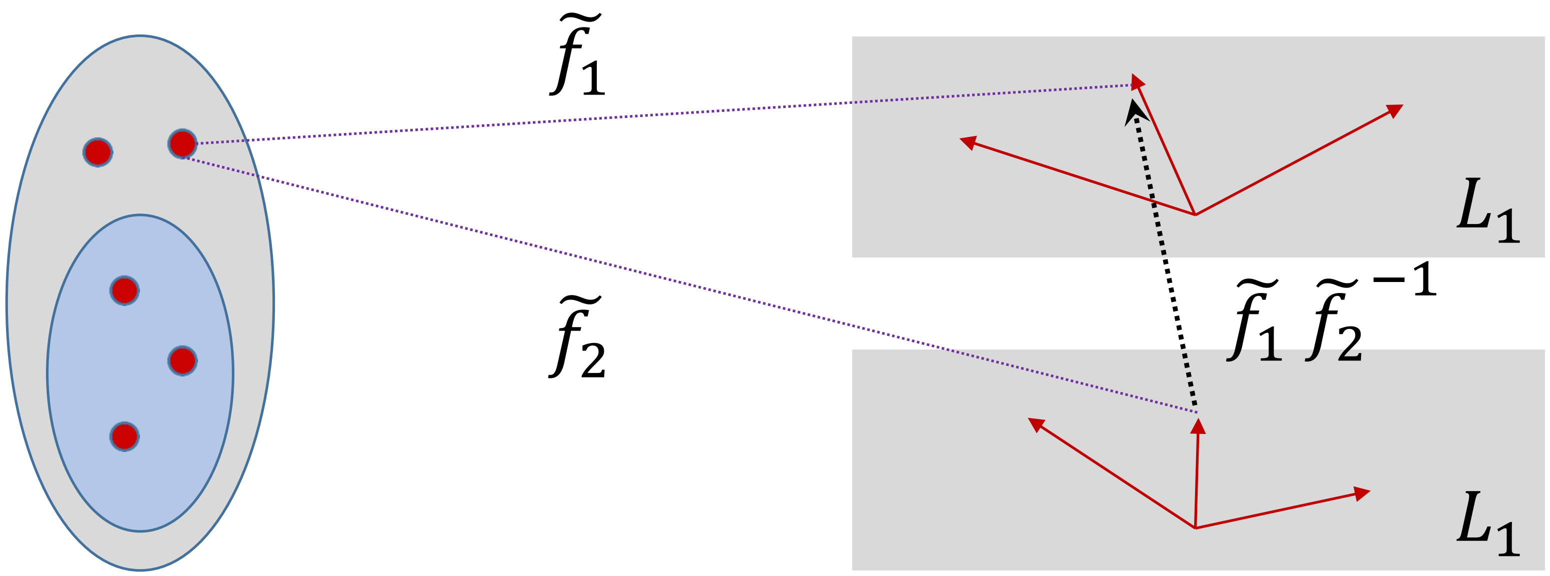


Bob: gives maps $f_1 : T \to L_1$ and $f_2 : T \to L_1$

Need:

$$\mathrm{bk}_G(\widetilde{f_1}) \le \mathrm{bk}_H(f_1)$$
$$\mathrm{bk}_H(f_2) \le Q\, \mathrm{bk}_G(\widetilde{f_2})$$

Relate $f_1 f_2^{-1}$ and $\widetilde{f_1}\,\widetilde{f_2}^{-1}$



Ball's Open Problem & Sparsification

[MM '10]

$$Q_k^{cut} = e_k(\ell_1, \ell_1) \le e_k(\ell_2, \ell_1) \cdot C\sqrt{\log k} \log\log k$$

here, $C\sqrt{\log k} \log\log k$ is the distortion of the Fréchet embedding of ℓ_1 into ℓ_2 by Arora, Lee, Naor '07.

If $e_k(\ell_2, \ell_1) \le C_{\text{Ball}}$ then

$$Q_k^{cut} \le C'\sqrt{\log k} \log\log k$$

Outer bi-Lipschitz extendibility

Bi-Lipschitz Kirszbraun Theorem?

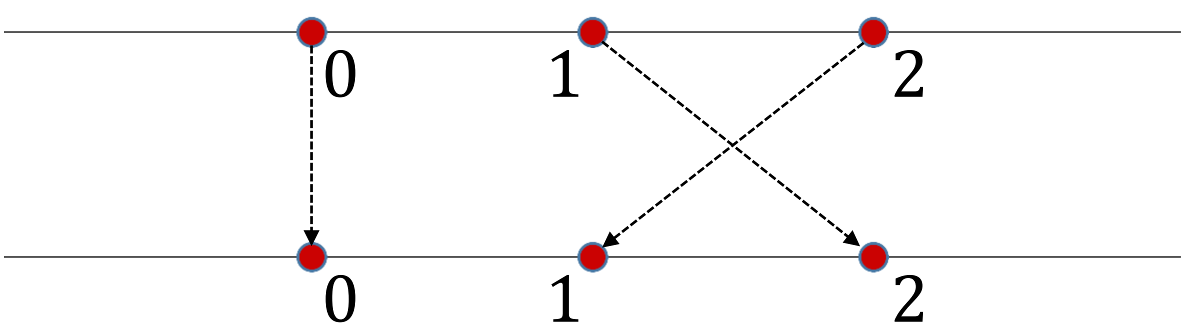
Let $A \subset \ell_2^m$. Can we extend a bi-Lipschitz map $f: A \to \ell_2^n$ to a bi-Lipschitz map $\tilde{f}: \ell_2^m \to \ell_2^n$?

No!

- $\mathbb{R}^2 \to \mathbb{R}$. Assume $\mathbb{R} \subset \mathbb{R}^2$. Extend $\mathrm{f} = id_{\mathbb{R}}$?

There is even no injective extension of f to $\mathbb{R}^2$.

- $\mathbb{R} \to \mathbb{R}$. f maps $0, 1, 2$ to $0, 2, 1$, respectively. There is even no continuous one-to-one extension.



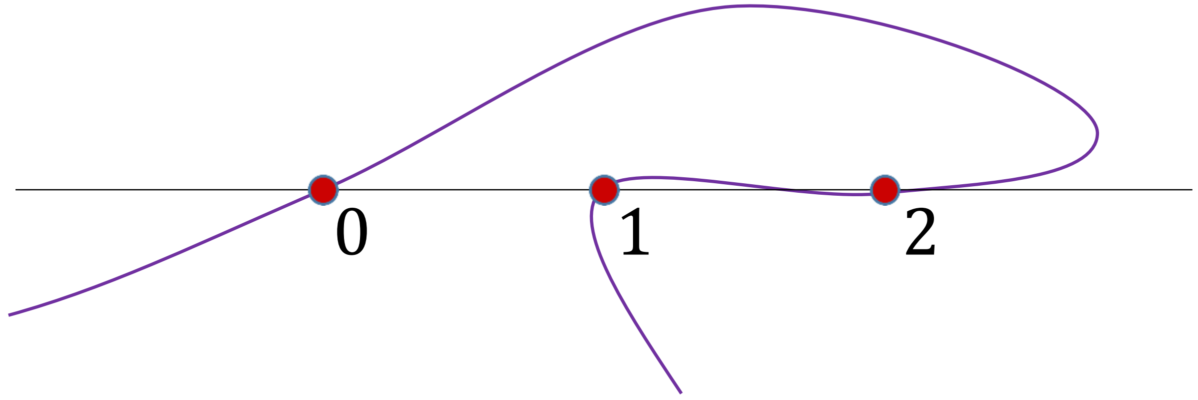
Bi-Lipschitz Kirszbraun Theorem?

Can overcome these obstacles by using extra dimensions!

• $\mathbb{R}^2 \to \mathbb{R}$. Assume $\mathbb{R} \subset \mathbb{R}^2$. Extend $\mathrm{f} = id_{\mathbb{R}}$?

Assume that target $\mathbb{R} \subset \mathbb{R}^2$. Then $\tilde{f} = id_{\mathbb{R}^2 \to \mathbb{R}^2}$

• $\mathbb{R} \to \mathbb{R}$. f maps $0, 1, 2$ to $0, 2, 1$, respectively.



Outer bi-Lipschitz extension

Given $A \subset \ell_2^m$ and bi-Lipschitz $f: A \to \ell_2^n$.
Assume $\ell_2^n \subset \ell_2^N$.

$\tilde{f}: \ell_2^m \to \ell_2^N$ is an *outer bi-Lipschitz* extension of f if

$$\tilde{f}(a) = f(a) \text{ for every } a \in A$$

and $\tilde{f}$ is bi-Lipschitz.

[MMMR '18] For every bi-Lipschitz $f: A \to \ell_2^n$, there exists an outer bi-Lipschitz extension with

$$D(\tilde{f}) \leq 3D(f).$$

Outer bi-Lipschitz extension

Near isometric case. What happens when

$$D(f) = 1 + \varepsilon\,?$$

If $\varepsilon = 0$, i.e., f is an isometric map, there is an isometric extension $\tilde{f}$

Is there a bi-Lipschitz extension with

$$D(\tilde{f}) = 1 + o(1) \text{ as } \varepsilon \to 0\,?$$

For $f : A \to \mathbb{R}, A \subset \mathbb{R}$, yes! There is an extension with

$$D(\tilde{f}) = 1 + \frac{1}{\log^2 1/\varepsilon}$$

The bound is tight.

Outer bi-Lipschitz extension

Near isometric case. What happens when

$$D(f) = 1 + \varepsilon\,?$$

If $\varepsilon = 0$, i.e., f is an isometric map, there is an isometric extension $\tilde{f}$

Is there a bi-Lipschitz extension with

$$D(\tilde{f}) = 1 + o(1) \text{ as } \varepsilon \to 0\,?$$

For a one-point extension of $f: A \to \ell_2^n$

$$D(\tilde{f}) = 1 + C\sqrt{\varepsilon}$$

The bound is tight.

Summary

Characterized the optimal distortion of cut and flow vertex sparsifiers in terms of Lipschitz extension constants.

- Find $e_k(\ell_1, \ell_1)$ and $e_k(\ell_\infty, \ell_\infty \oplus_1 \dots \oplus_1 \ell_\infty)$
- Is $e_k(\ell_2, \ell_1) < \infty$?

Defined outer bi-Lipschitz extension and proved an analogue of Kirzsbraun theorem for it. Partial results for nearly isometric maps.

- Understand the nearly isometric case.

Applications to a prioritized dimension reduction.